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OF NOTED MATHEMATICIANS, ETC.

EDITOR:

B. F. FINKEL, A. M., M. Sc.,

MEMBER OF THE LONDON MATHEMATICAL SOCIETY; MEMBER OF THE AMERICAN MATHEMATICAL SOCIETY, AND PROFESSOR OF MATHEMATICS AND PHYSICS IN DRURY COLLEGE, SPRINGFIELD, MISSOURI.

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HOPKINS, G. I., Instructor in Mathematics and Physics in High School, Manchester, N. H. (40, 75)
HOWIE, JOHN M., Professor of Mathematics, The Nebraska State Normal, Peru, Neb. (56, 104)
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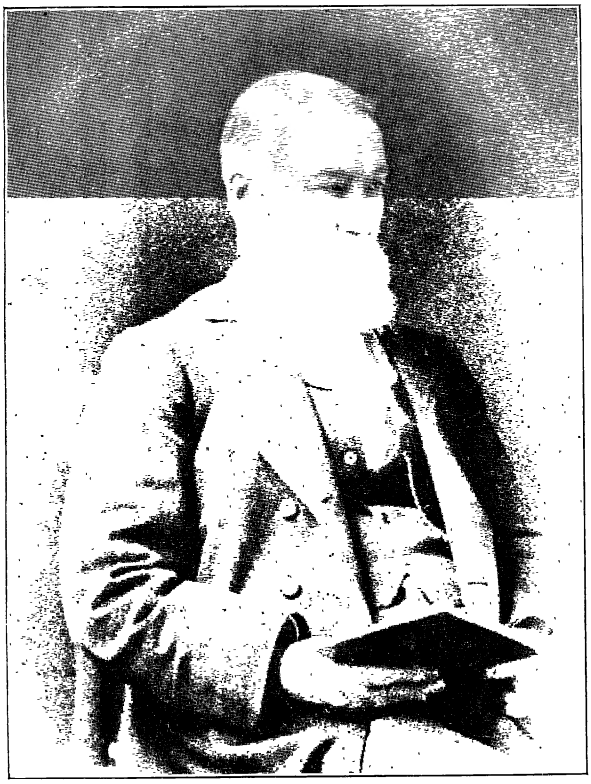

ROBERT TUCKER.

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THE THEORY OF MATHEMATICAL INFERENCE.

By G. J. STOKES, Professor of Logic and Philosophy in Queen's College, Cork, Ireland.

One of the simplest theories of mathematical inference and by no means the least plausible is that involved in the extreme Nominalist doctrine that the conclusion of a train of reasoning is but a restatement in changed language of the original data from which we start, a renaming in short, of what had been otherwise expressed. Nowhere has this theory as applied to mathematical inference been more clearly put forward than by James Mill in his "Analysis of the Phenomena of the Human Mind."

"The Predications of Arithmetic" he says are another instance of the same thing. 'One and one are two.' This again is a mere process of naming. What I call one and one, in numbering things, are objects, sensations, or clusters of sensations; suppose, the striking of the clock. The same sounds which I call one and one I call also two; I have for these sensations, therefore, two names which are exactly equivalent: so when I say, one and one and one are three: or when I say, two and two are four: ten and ten are twenty: and the same when I put together any two numbers whatsoever. The series of thoughts in these instances is merely a series of names applicable to the same thing, and meaning the same thing.

Besides the two purposes of language, of which I took notice at the beginning of this inquiry; the recording of a man's thoughts for his own use, and the communication of them to others; there is a use, to which language is subservient, of which some account is yet to be given. These are complex sensations,

and complex ideas, made up of so many items, that one is not distinguishable from another. Thus, a figure of one hundred sides, is not distinguishable from one of ninety-nine. A thousand men in a crowd are not distinguishable from nine hundred and ninety-nine. But in all cases, in which the complexity of the idea arises from the repetition of the same idea, names can be invented upon a plan, which shall render them distinct, up to the very highest degree of complication. Numbers are a set of names contrived upon this plan, and for this very purpose. Ten and the numbers below ten, are the repetition of so many ones: twenty, thirty, forty, etc., up to a hundred, are the repetition of so many tens: two hundred, three hundred, etc., the repetition of so many hundreds, and so on. These are names, which afford an immediate reference to the ones or units, of which they are composed; and the highest numbers are as easily distinguished by the difference of a unit as the lowest. All the processes of Arithmetic are only so many contrivances to substitute a distinct name for an indistinct one. What, for example, is the purpose of addition? Suppose I have six numbers, of which I desire to take the sum, 18, 14, 9, 25, 19, 15; these names, eighteen, and fourteen, and nine, etc., form a compound name; but a name which is not distinct. By summing them up, I get another name, exactly equivalent, one hundred, which is in the highest degree distinct, and gives me an immediate reference to the units or items of which it is composed; and this is of the highest utility.

That the Predications of Geometry are of the same nature with those of Arithmetic, is a truth of the greatest importance, and capable of being established by very obvious reasoning. It is well known, that all reasoning about quantity can be expressed in the form of algebraic equations. But the two sides of an algebraic equation are of necessity two marks or two names for the same thing; of which the one on the right-hand side is more distinct, at least to the present purpose of the inquirer, than the one on the left-hand side; and the whole purpose of an algebraic investigation, which is a mere series of changes of names, is to obtain, at last, a distinct name, a name the marking power of which is perfectly known to us, on the right-hand side of the equation. The language of geometry itself, in the more simple cases, makes manifest the same observation. The amount of the three angles of a triangle, is twice a right angle. I arrive at this conclusion, as it is called, by a process of reasoning: that is to say, I find out a name "twice a right angle," which much more distinctly points out to me a certain quantity, than my first name, "amount of the three angles of a triangle;" and the process by which I arrive at this name is a successive change of names, and nothing more; as any one may prove to himself by merely observing the steps of the demonstration."

It is easy to criticize the doctrine of this passage. Strictly taken it would reduce all mathematical reasoning to a series of identities—a mere change of names. If we admit however that total and partial identities—relations of wholes and parts—may be expressed by such changes, the doctrine coincides with the opinion of those who consider that the fundamental truths of mathemat-

ics are of an analytical not a synthetical character. This opinion has not only received the support of Leibnitz as well as of the school of philosophy to whom the truths of mathematics—especially those of Arithmetic and Algebra—represent merely verbal propositions, but it has been peculiarly strengthened by recent mathematical developments. If we consult the writings of those who have maintained the opposite doctrine, viz., the synthetical character of mathematical truths we shall find that they maintain this only of certain of the fundamental truths employed in mathematics. Thus the axioms proper of Euclid are admitted by Kant and Mansel to be analytical. When they seek for an example of a synthetical truth, they find it in the fifth and sixth postulates, sometimes enumerated as the eleventh and twelfth axioms. But it is precisely such principles as the latter which are now regarded as expressing not universal and necessary principles of all geometry but only the particular and contingent properties of the space with which we are acquainted ; and of which Clifford asserts "for all we know, any or all of them may be false." If now by mathematical inference or reasoning be understood the form of reasoning common to all mathematical thought, there seems to be left as a residuum, only those processes of analytical inference which are expounded in the ordinary formal or Aristotelian logic.

It is possible to support this view, to regard those principles which are really synthetic and fertile in mathematics, as either gathered from actual experience, or as hypothetically assumed in regard to some possible experience ; and, on the other hand to regard the process by which these fundamental assumptions are worked out into their consequences as purely syllogistic. The necessity with which these consequences flowed would then be strictly formal and logical. This view would appear to be in exact accordance with the general principles laid down by J. S. Mill on the subject of demonstration and necessary truths (Logic, Book II, Chapter VI). It differs from Mill's view, only in not regarding the axioms proper as inductions from experience, and in extending the postulates to embrace those possibilities of Non-Euclidean geometry which Mill did not contemplate. Notwithstanding some criticism of Mill, this I understand to be the view actually adopted by Clifford. A theory more completely agreeing with Mill's principles is put forward by Erdmann in his work 'Die Axiome der Geometrie' resting his conclusions on the investigations of Riemann and Helmholtz.

It is to be observed however that this doctrine is in complete opposition to the theory contained in the quotation from James Mill with which we started. On that theory the rich content of mathematics could be evolved by a series of analytical or verbal transformations. J. S. Mill in the chapter of the Logic from which we have quoted, clearly shows the impossibility of such a view. Mathematical inference leads to new truths, and new truths, according to Mill, can only be reached if inference be impregnated by experience. Hence Mill held that axioms and postulates come from this source.

It still remains to be asked, Can Formal Logic, Syllogism or mere verbal inference, perform the attenuated task left to it, viz., the inferring process ?

Can verbal propositions, if no longer competent to give a new content, nevertheless be the means of passing from one content to another? At first sight, the great resemblance between the elementary propositions of mathematics and the propositions of Formal Logic, seems to favour this view. When in Arithmetic we form the judgment $2+1=3$, or in Geometry discover the quantitative equivalence of the three angles of a triangle to two right angles, the resemblance to such purely analytical relations as are formulated by the *dictum de omni et nullo* is very great. In fact, our theory reduces predication itself to the equation of groups. Nevertheless, this resemblance is, I believe illusory, and so far is mere verbal inference from being able to perform the function, which James Mill allotted to it, that it cannot even perform the more modest task reserved to it by J. S. Mill.

It has long been a matter of observation that inferences exist, which while perfectly rigorous, yet do not admit of syllogistic analysis. The argument *a fortiori* is an example, and many others are furnished by what has been called the logic of relatives. These are examples of what older logicians called material consequence. The device by which Mansel and others have tried to reduce them to syllogistic form, is, as De Morgan truly says, an evasion. Both kinds of inference are found in mathematics but that inference which is most peculiarly mathematical is of the second kind, which escapes or defies the analysis of Formal Logic. Mill has exhibited Euclid I, 5 in syllogistic form but the reduction of the reasoning to this form is purely external. The force of mathematical reasoning is independent of the reduction. This is already implied in Dugald Stewart's remark, referred to by Mill, that it is not necessary to our seeing the conclusiveness of the proof in mathematical reasonings, that the axioms should be expressly adverted to. The same thing is conceded, perhaps unwittingly by Hamilton. "Mathematical, like all other reasoning," he says, "is syllogistic; but here *the perspicuous necessity of the matter necessitates the correctness of the form:* we cannot reason wrong."

If, now, we have reason to believe, that there exists in mathematical inference, a "necessity of the matter" existing in itself, and not merely derivative from the logical form, the question arises: Can we isolate it for itself? If we can do this we shall have grounds for concluding, that what is really fruitful in mathematics, is not, as has been so often supposed, initial definitions, axioms, etc., which we afterwards logically analyze and develop, but a certain synthetic mode of inference not identical with the analytical inference of formal logic, but distinct from it, *sui generis*, and perhaps opposed in character.

Before passing to the consideration of this synthetic and material necessity, it may be well to point out, that the whole controversy as it has hitherto existed between the *a priori* and *a posteriori* schools, between the Kantian and the empiricist, becomes for us irrelevant. In a paper in Mind in 1884 I pointed out that the Kantian theory does not explain the synthesis in mathermatical truths. It only places the synthesis finished and complete, in the subject. Clifford puts the same point in another way when he says that the Kantian theory

makes the general statements of mathematics into particular statements. On the other hand the empirical school makes no attempt to explain the necessity of the consequence in itself. It is this which we are seeking to isolate. When this isolation has been effected, it will appear that mathematical inference is so related to ordinary analytical reasoning, as to stand in need neither of the Kantian nor experience hypothesis, neither of *a priori* form nor of inseparable association, in order that its necessary and universal character may be accounted for.

The idea has sometimes been entertained, that logical analysis has reached through the calculus contained in Boole's Laws of Thought this more penetrating character and that here we have an instrument which can make inferences beyond the range of formal logic. This is not so. Mr. A. J. Ellis has shown that it really does less; and in the papers of Leslie Ellis it is pointed out that for dealing with those forms of collateral inferences which we find in tha logic of relatives it is as ineffectual as formal logic.

The immense suggestiveness of Boole's work lies in the circumstance that he has reduced formal logic to a calculus and that logical doctrines are put in a form in which they suggest mathematical analogues. But the suggestion is, for the most part one of opposition. The things are brought into the same plane and thereby their opposition becomes apparent. This is precisely what from the foregoing we should expect. If, now, taking Boole's Laws of Thought as an exposition of formal logic in mathematical form, we ask the question 'Can we find within the area of mathematics any calculus presenting that antithetic but complementary character which a form of synthetic inference should present, as contrasted with a form of analytical inference?' I think we may answer 'Yes.' If we compare the fundamental equations of Boole's Laws of Thought with the equations which characterize some of those forms of multiplication discussed by Grassmann, and employed by him in his 'Ausdehnungslehre' we seem to find the antithesis which we seek. Already in the principles of the Differential Calculus this antithesis was to be found, and by an intuition of genius was perceived by Boole. For the purpose of this paper we shall confine ourselves to the following equations.

In Boole's Laws of Thought we find

$$x^2 = x \qquad (1).$$

$$x(1-x) = 0 \qquad (2).$$

In Grassmann's we find

$$e_p{}^2 = 0 \qquad (3).$$

$$e_p e_s = -e_s e_p \qquad (4).$$

If we compare these two sets of equations we shall find that they differ in this fundamental characteristic, that whereas the first set of equations belong to an algebra of self-identical unrelated units the second set belong to an algebra in which relation, synthesis, references beyond self, is essential.

The equations (3) and (4) have given rise to some difference of opinion in

regard to their mutual relations. H. Hankel, H. J. S. Smith and Mr. Whitehead assert that the first of these equations follows necessarily from the second. Buchheim however points out that this is not the case, that (4) does not involve (3), and Clifford regards (4) as following from (3); that is, we may assume (4), without asserting (3), but not (3) without (4).

In this Clifford and Buchheim seem to be right. The equation $r_s r_p = -r_p r_s$ proves $r_s^2=0$ (in the case where $p=s$) if r_s and r_p exist only as necessary relatives which become equal to 0 when the difference and therefore the relation between them disappears. If they exist as such. equation (4) seems to follow, but equation (3) follows from this relativity, not from (4), which may be affirmed even when the factors involved exist out of and apart from their mutual relation. The truth seems to be that equations of this type of algebra may participate in characters derived from equations of the Boole type, and so give rise to hybrid species which may be useful for particular interpretations. In general we may distinguish (1) the purely logical algebra whose relations only of identity and difference, coincidence and exclusion are admitted. (2) The algebra of material consequence where the units are not indifferent but where the *entia* involved are essentially relative and the algebra consequently synthetic.

Both these algebras lead to a final equation in which each passes over into the opposite. Boole's $x(1-x)=0$ is in his system the ultimate condition of logical interpretability but it also expresses, the one relation into which logical terms enter with each other, and the equation $i_s^2=\pm 1$ (Clifford) expresses the disappearance of relativity and the return of the merely logical relations of difference and identity.

Ordinary Arithmetic, Algebra, and Metrical Geometry are a combination of these two kinds of Algebra. Up to a certain point, the principles of both can be alternately applied, but at one point the necessity arises for a deeper fusion. The introduction of relation into a logical calculus involves, as has been pointed out by Mr. Venn, the very thing which Boole excludes—the admission of exponents. Conversely, in ordinary mathematics the appearance of exponents involves essential relation. In the sign $\pm\sqrt{}$ the alternation of the purely logical and the relational aspects is still continued, and the same is the case in the Differential Calculus. But in the latter, and in imaginary expressions this alternation of independent aspects ceases. It the calculus the externality of the logical consideration ceases at infinity. In the imaginary it ceases in the immediate combination of the signs $+$ and $-$.

In a paper on the Imaginary of Logic (British Association, 1898) I put forward the view that as the square root of a positive quantity is $+$ *or* $-$ the square root of a negative quantity may be expected to be $+$ *and* $-$, in view of the logical relation between '*and*' and '*or*' pointed out originally by De Morgan, and subsequently, and independently by Schroeder. This theory is the opposite of one put forward in an early number of the Cambridge and Dublin Mathematical Journal by Gregory, viz., That the signs $+$ and $-$ are themselves the subject of the exponential operation. The object of the paper was to show that

a necessary relation of the signs $+$ and $-$ as affecting the factors respectively was the essential characteristic of imaginary quantities. The essence of this relation would then coincide with Boole's $x(1-x)$. The subsequent portion of the paper sought to verify this theory throughout the various geometrical interpretations which imaginaries have received.

Since the paper was written I have found that its conclusions receive support from a remarkable series of papers by Mr. A. B. Kempe F. R. S. Mr. Kempe has shown that between the mathematical theory of points and the logical theory of statements, a striking correspondence exists. Between the laws defining the form of a system of points, and those defining the form of a system of statements, perfect sameness exists with one exception. The former is subject to a law to which the latter is not subject. It is sufficient here to say that it is the law "which expresses the fact that two straight lines can only cut once."

From these conclusions we may draw the converse inference, that the laws which govern geometrical theory can be deduced from logical or purely analytical principles, taken in conjunction with that law in which the form of a system of points differs from the form of a system of statements. We have now to ask, Is there anything omitted from the form of a system of statements as contemplated by Mr. Kempe, or by the ordinary logic (and there is complete agreement between them) which would account for the absence of the particular law which distinguishes geometrical theory? I think there is. Mr. Kempe in order to effect his assimilation of the logical to the geometrical theory, and in particular in explaining the processes of immediate inference has introduced two constants which play the same part in the logical theory that the 'absolute' does in geometry. He entitles them 'truism' and 'falsism' respectively. It is by relation to these that such logical relations as contrariety, sub-contrariety, sub-alternation analogous to the metrical relations of points in geometry are determined. He considers "truisms" and "falsisms" as propositions or statements standing in the system of statements on the same footing with all other statements. In reality this is not so. The truism and falsism of Mr. Kempe are really the laws of Identity and Contradiction in disguise, and every synthetic statement or proposition expresses more than what these laws require. The principle that a real proposition refers to, or is a synthesis with, something more than itself, is as old as Aquinas, and is indeed the fundamental principle which makes our thinking dependent on experience (Cf. Bradley's Principles of Logic). It is the non-recognition of this which prevents Mr. Kempe from evolving the relation of non-collinearity from the relation of a truism and falsism to each other which ought to be capable of being done, if it were true that these propositions could rank *pari passu* with all other propositions. A truism is not as such a true proposition. Apart from the postulate of synthesis no logical relation exists between the truism and falsism. Contradictories are in this case compatible as Venn and Kant before him have pointed out.

If these views be true I believe it to be possible to deduce the properties of Euclidean space, not from the analytical laws of thought, but from the pure

postulate of synthesis, when subjected to conditions arising from these laws. The postulate can be shown to involve two things (1) Infinity, (2) the necessary relation or connection of what Mr. Kempe styles truism and falsism equivalent to Boole's $x(1-x)=0$.

It remains to point out the connection which exists between the logical 'absolute' of Mr. Kempe and the theory of the imaginary referred to in the course of this paper. I was led to that analysis from consideration of the correspondence between the logical relation of a copulative or conjunctive to a disjunctive proposition, and the mathematical relation of imaginary roots to the roots of positive quantities. A similar relation has been perceived by Mr. Kempe. "The symmetrical resultant of the triad [of statements] a, b, f [f=falsism] is the statement usually written, a and b, and the symmetrical resultant of the triad, a, b, t [t=truism] is the statement usually written a or b." If the relation of truism and falsism or in Boole's language $x(1-x)$ be, as we assert the essence of the mathematical imaginary; and if the same constants have in Mr. Kempe's analysis disclosed themselves as the essence of the geemetrical 'absolute' a deep-lying relation is revealed between the methods of metrical and projective Geometry.

Finally, there exists a curious analogy between the geometrical, and certain theories of the metaphysical absolute. The temptation lies near at hand to evolve the synthetical or given element out of the laws of Identity and Contradiction. Fichte's evolution of the Non-Ego out of the Ego is effected in this way and thus arises his theory of the absolute Ego. Precisely the some error is committed, if an attempt be made to dispense with the postulate of synthesis, with the *given*, and to evolve mathematics out of analytical propositions. Mr. Kempe comes near this mistake when he treats truisms and falsisms as propositions on a line with all other propositions.

It remains to draw the final conclusions of this paper. The fertile propositions of mathematics from which its wealth of content and the treasures of mathematical knowledge are drawn, are not synthetical in the sense in which Kant and the Empiricists alike maintain them to be, viz., that the truths pre-exist and are thus seen to be synthetical, the synthetical character being as it were something subsequent to the content of the proposition and attaching to it as it were adjectivally; but in this sense that those propositions are themselves the product of pure synthesis, that the very possibility of advance from entity to entity or unit to unit, or relation to correlate, determines all those laws which mathematics is employed in exploring and tracing into all their consequences, and which are infinitely more fruitful than the analytical laws of Formal Logic or the Calculus of classes and statements. Pure synthesis generally is that "necessity of the matter" of which Hamilton spoke, the principal of material consequences, which characterizes every genuine department of mathematics and defies further logical analysis.

Fancourt, Balbriggan, Ireland, 10 August, 1899.

EXAMPLES OF A FEW ELEMENTARY GROUPS.*

BY DR. G. A. MILLER.

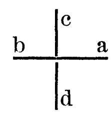
It is easy to verify that the following four substitutions,

$$1 \qquad ab \qquad cd \qquad ab.cd$$

constitute a substitution group, which has the following properties: (1) The product of any two of these substitutions of order two is the third, and the product of the three is identity; (2) These products are independent of the order of the factors, *i. e.*, all the substitutions are commutative;† (3) The smallest group that contains two of these substitutions of order two must also contain the third, *i. e.*, the group is generated by any two of its substitutions of order two, but it is not generated by one of them; (4) The group contains three subgroups of order two and one of order one.

We proceed to give several geometric illustrations of this group.

Representing the positive half of the x-axis by a, the negative half by b, the positive half of the y-axis by c, and the negative half by d, we observe that the rotation of the plane around the y-axis through an angle of 180° corresponds to the substitution ab, and the rotation around the x-axis through 180° corresponds to cd. The effect obtained by these two rotations, in succession, is clearly equivalent to a rotation of the plane through 180° on the origin as a pivot. This operation is also of order two and it corresponds to $ab.cd$.

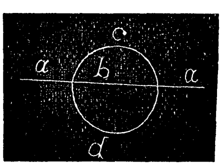


Since the law of combination of these three operations is exactly the same as that of the given substitutions we say that these three operations and identity constitute a group which is *simply isomorphic* to the given substitution group. Representing the given operations analytically we obtain the following equations:

$$ab\begin{cases} x'=-x \\ y'=y \end{cases} \qquad cd\begin{cases} x'=x \\ y'=-y \end{cases} \qquad ab.cd\begin{cases} x'=-x \\ y'=-y \end{cases}$$

It may be observed that the third of these operations is equivalent to a rotation of space around the z-axis through 180°

We may obtain another geometric illustration of the same group by considering the inversion of the plane with respect to any circle of radius k and the rotation of the plane through an angle of 180° around a line passing through the



*In this article we shall not presuppose any knowledge of the theory of group3 except the facts which were developed in our article in the November number of this Journal.

†If every two substitutious of a group are commutative the group is said to be Abelian. Hence this group of order four is an Abelian group. The group of order six which is given in the said article of the November number of this Journal is non-Abelian.

center of the circle. Since each of these operations is of order two and since they are commutative their product must be of order two and it must be commutative with each of these two operations. For, if s_1, s_2 represent two different commutative operations of order two we have

$$(s_1 s_2)^2 = s_1 s_2 s_1 s_2 = s_1^2 s_2^2 = 1.$$

This proves that $s_1 s_2$ is of order two. From $s_1 s_2 s_1 = s_1 s_1 s_2$ we observe that s_1 is commutative with $s_1 s_2$. Similarly we see that s_2 is commutative with $s_1 s_2$. Hence we observe that *any two different commutative operations of order two must generate a group which is simply isomorphic to the given group of order four.* If we use the given line as the x-axis and the perpendicular to it through the center of the circle as the y-axis we may represent the given operations analytically as follows:

$$*\, ab\begin{cases} x' = \dfrac{k^2 x}{x^2 + y^2} \\ y' = \dfrac{k^2 y}{x^2 + y^2} \end{cases} \qquad cd\begin{cases} x' = x \\ y' = -y \end{cases} \qquad ab.cd\begin{cases} x' = \dfrac{k^2 x}{x^2 + y^2} \\ y' = -\dfrac{k^2 y}{x^2 + y^2} \end{cases}$$

To verify analytically that the last one of these operations is of order two we let

$$x'' = \frac{k^2 x'}{x'^2 + y'^2} = k^4 \frac{x}{x^2 + y^2} \frac{(x^2 + y^2)^2}{k^4 (x^2 + y^2)} = x,$$

$$y'' = -\frac{k^2 y'}{x'^2 + y'^2} = k^4 \frac{y}{x^2 + y^2} \frac{(x^2 + y^2)^2}{k^4 (x^2 + y^2)} = y.$$

We have now given two geometric illustrations of the given group of order four† and we observed that the characteristic property of this group is, that it is generated by two different commutative operations of order two. There is another substitution group of order four whose characteristic property is entirely different. The substitutions of this group are

$$1 \qquad ac.bd \qquad abcd \qquad adcb.$$

Each of the last two substitutions is the third power of the other and each one of these generates the entire group; *i. e.*, the smallest group that contains one of these substitutions must contain all the substitutions of this group of order four. It can readily be verified that these four substitutions obey the same com-

*These substitutions may be obtained by representing the segment of the axis which is outside the circle by a, the segment within the circle by b, the upper semi-circumference by c, and the lower semi-circumference by d.

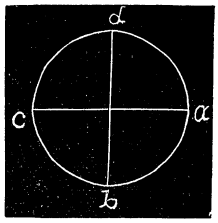
†In the latter example we could have inverted space with respect to a sphere intead of inverting the plane with respect to a circle.

binatory laws as the numbers which are written below them in the following arrangement:

$$\begin{array}{cccc} 1 & ac.bd & abcd & adcb \\ 1 & -1 & \sqrt{-1} & -\sqrt{-1} \end{array}$$

The last two numbers can evidently be interchanged without affecting the laws of combination, but none of the other numbers permit such an interchange.

If we denote the points where two perpendicular diameters meet a circle by a, b, c, d, we observe that the substitution $abcd$ is equivalent to rotating this circle on its center through 90°, $ac.bd$ is equivalent to a rotation through 180°, and $adcb$ is equivalent to a rotation through 270°, or through $-90°$. The characteristic property of this group is that it is generated by an operator* of order four. When a group is generated by a single operator of order n it is called the *cyclical* group of order n. It should be observed that the cyclical group of order four contains only one subgroup of order two, viz., the one which corresponds to the rotations through 180° and 360°, while the given non-cyclical group of this order contains three such subgroups.



We have now considered two groups of order four whose combinatory laws are different, *i. e.*, two groups which are not simply isomorphic. Such groups are said to be distinct *abstract* groups. Two groups which are simply isomorphic are said to be the same abstract group, regardless of the notation by means of which they may be represented, *e. g.*, 1, ab, cd, $ab.cd$, and 1, $ab.cd$, $ac.bd$, $ad.bc$ are different as substitution groups but they represent the same abstract group since the law of combination of their substitutions is the same. We may state without proof that there are only two abstract groups of order four; *i. e.*, If four operators form a group their laws of combination must be the same as those of one of the given groups of order four.

We proceed to give a geometric illustration of the group of order six which is composed of all the substitutions that can be formed with three letters.† The substitutions of this group are

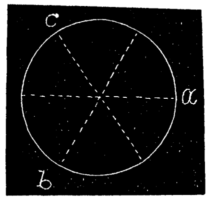
$$1 \qquad abc \qquad acb \qquad ab \qquad ac \qquad bc$$

Dividing the circle into three equal parts and drawing diameters through these points of division, we observe that abc and acb correspond to rotations of the circle on its center through 120° and 240°, respectively; ab, ac, and bc corres-

*The substitutions of a group represent operators as well as the result of operations. The group elements may therefore be called operations or operators. It is necessay to distinguish between the group elements and the elements of the substitutions, the former term is frequently used to denote the operators of a group since the group is really composed of these operators as elementary parts. When the word "element" is used in connection with a group we have sometimes to decide from the context whether it means an operator or a letter of the substitutions of the group.

†The group which is composed of all the possible substitutions of degree n is called the *symmetric* group of degree n. It is of order $n!$. Cf. THE AMERICAN MATHEMATICAL MONTHLY, Vol. VI, page 257.

pond to the rotations of the plane through 180° around the diameters going through c, b, and a, respectively. If we perform any two of these rotations in succession the result is equivalent to a single rotation which corresponds to the product of the two substitutions corresponding to the two rotations and taken in the same order ; *e. g.*, the rotation on the center through 240°, followed by rotation through 180° around the diameter through a is equivalent to the single rotation through 180° on the diameter through c, since $acb.bc=ab$. This result should also be seen geometrically.



All the products given in Vol. VI, page 256, of this Journal, may be directly verified by means of the last figure. The six rotations which correspond to the substitutions of the symmetric group of degree three are thus seen to form a very interesting group of rotations according to which the plane (or space) may be transformed. The determination of all the possible groups of motion by means of which space may be transformed forms a very interesting problem in the theory of groups, which was first studied by Camille Jordan, *Annali di Matematiche*, 1868, Vol. 2, page 167. The group of finite rotations are given in somewhat greater details in Klein's *Ikosaeder*, 1884, Chapter I.

Another important illustration of the symmetric group of three elements is furnished by the six anharmonic ratios of four points. These ratios may be placed in six different ways in a 1, 1 correspondence with the substitutions of this symmetric group. One of these ways is as follows :

1	abc	acb	ab	ac	bc
λ, λ	$\lambda, \frac{\lambda-1}{\lambda}$	$\lambda, \frac{1}{1-\lambda}$	$\lambda, 1-\lambda$*	$\lambda, \frac{1}{\lambda}$	$\lambda, \frac{\lambda}{\lambda-1}$

where the notation $\lambda, \frac{\lambda-1}{\lambda}$ means that λ is to be replaced by $\frac{\lambda-1}{\lambda}$. *E. g.*, performing the third and fourth operation in succession, we have $\lambda, \frac{1}{1-(1-\lambda)}$ $=\lambda, 1/\lambda$ just as $acb.ab=ac$; performing the fourth and third operation in succession, we have $\lambda, 1-\frac{1}{1-\lambda}=\lambda, \frac{\lambda}{\lambda-1}$ just as $ab.acb=bc$, etc. For other illustrations of this group the reader may consult Burnside's *Theory of Groups*, 1897, page 18.

The symmetric group of degree four contains 24 substitutions. These correspond to the 24 rotations which transform a cube into itself, for these rotations permute the four diagonals of the cube in every possible manner. The axes of rotation are the lines which join the middle points of the opposite faces, those which join the middle points of the opposite edges, and the diagonals. There are three axes of the first kind and we may rotate the cube around one of these

*It may be observed that $(\lambda, 1-\lambda)^2=1$ while $(\lambda, \lambda-1)^\alpha=\lambda, \lambda-\alpha$; *i. e.*, the first of these two substitutions is of order two while the second does not have a finite order.

axes through an angle of 90°, 180°, 270°, or 360° so that after each rotation the entire cube occupies the same space as it did before the rotation. Hence the symmetric group of degree four contains three cyclical subgroups of order four. Each of these contains a subgroup of order two and no two of these subgroups of order two are identical.

There are six axes of the second kind and we may rotate the cube into itself around one of these axes through 180° or 360°. As the corresponding subgroups of order two are different from the three given above we observe that the symmetric group of degree four contains nine subgroups of order two. The rotations around the diagonals correspond to the four snbgroups of order three that are contained in the symmetric group of degree four. We have now employed all the possible rotations which transform the cube into itself without changing its center, and have seen that the corresponding permutations of the diagonals give all the possible substitutions that can be formed with four elements. These 24 rotations constitute an interesting group of motion.

It is easy to see that the different powers of a circular substitution of degree n $(a_1 a_2 a_3 \ldots\ldots a_{n-1} a_n)$ constitute a group of order n. When $n=3$ we have the substitutions 1, $a_1 a_2 a_3$, $a_1 a_3 a_2$, and when $n=4$, the substitutions 1, $a_1 a_3$. $a_2 a_4$, $a_1 a_2 a_3 a_4$, $a_1 a_4 a_3 a_2$. These groups have been considered. In general we may divide the circumference of a circle into n equal parts and represent the points of division by $a_1, a_2, a_3, \ldots \ldots, a_n$. The n different positive rotations around the center of the circle through angles which are divisible by $2\pi/n$ will clearly constitute a group of operations that is simply isomorphic to the substitution group generated by the given circular substitution. Since the equation $x^n - 1 = 0$ has primitive roots all the roots of this equation constitute a group which is simply isomorphic with the cyclical group of order n.

From the preceding examples it may be inferred that the same group may present itself in many different forms as well as in different branches of mathematics. The fundamental group concept is that there is a system of operations (substitutions, rotations, complex numbers, etc.) such that the product of any two of them and the square of any one are again in the system. This necessary condition is not always a sufficient condition that a system of operations may constitute a group, but many operations, such as substitutions, obey *per se* the other necessary conditions.*

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*Cf. Burnside, *Theory of Groups of a Finite Order*, page 11, or Weber's *Lehrbuch der Algebra*, Vol. 2, page 2.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and the PROPOSER.

First, put in two layers of 144 balls each, then fourteen layers of 121 and 144 balls in alternate layers.

This gives nine layers of 144 each=1296
seven layers of 121 each= 847

Total=2143

Every four balls in 144 layer makes with each ball in a 121 layer a square pyramid, base 1 inch, and slant edge 1 inch.

$\therefore$ Altitude$=\frac{1}{2}\sqrt{2}=.7071$ inches.

$.7071\times14=9.8994$ inches. $9.8994+2=11.8994$ inches.

$12-11.8994=.1006$ inch to spare.

Also solved by *CHARLES C. CROSS.*

121. Proposed by PAUL ROULET, A. M., Professor of Mathematics, Fairmount College, Wichita, Kas.

Three men, named Adams, Morris, and Stoughton, with their sons, Edward, Nathan, and Walter, have each a piece of land in the form of a square. Mr. Adams' piece is 23 rods longer on each side than Nathan's, and Mr. Stoughton's piece is 11 rods longer on each side than Edward's. Each father has 63 square rods of land more than his son. Which of these persons is father and son, respectively?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let the initial letter of person's name represent the length of the respective person's square piece of land.

Then $63+E^2=\square$, $63+N^2=\square$, and $63+W^2=\square$.

We are now to find which of these squares equals, respectively, A^2, M^2, and S^2.

The basis of work is the identity,

$$mn+\left(\frac{m-n}{2}\right)^2=\left(\frac{m+n}{2}\right)^2,$$

where $mn=63=9\times7=21\times3=63\times1$.

From these sets of factors of 63, we obtain

$$63+1^2=8^2,\ 63+9^2=12^2, \text{ and } 63+31^2=32^2.$$

Hence 1^2, 9^2, and 31^2 are the pieces of land belonging to the sons, and 8^2, 12^2, and 32^2 those belonging to their respective fathers.

But, from conditions of problem, we have $A-23=N$, and $S-11=E$.

Whence, it is evident that A must equal 32, and S must equal 12, from which we find $N=9$, and $E=1$.

Therefore, of the two remaining values, $M=3$, and $W=31$.

Now, since each father has 63 square rods of land more than his son, the man having 8^2 is the father of the son having 1^2; the man having 12^2 is the father of the son having 9^2; and so on.

$\therefore$ Mr. Morris is Edward's father; Mr. Stoughton is Nathan's father; and Mr. Adams is Walter's father.

A similar problem, said to be published in several newspapers of recent issue, is as follows: "Three Dutchmen, Hans, Klaus, and Hendricks, went to market to buy hogs, and took their wives with them. The names of the wives were Gertrude, Anna, and Katrine; but it was not known which was the wife of each man. They each, men and wives, bought as many hogs as each paid shillings, respectively, for each hog; and each man spent three guineas more than his wife. Hendricks bought 23 hogs more than Gertrude, and Klaus bought 11 more than Katrine. What was the name of each man's wife?"

A solution, similar to the above, gives Katrine as Hans's wife, Gertrude as Klaus's wife, and Anna as Hendricks's wife.

II. Solution by **W. H. CARTER**, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

It is shown in geometry that the difference between two square areas is equal to a rectangle whose base is the sum of the sides of the squares and whose altitude is the difference between the sides.

63 square rods is the area of this rectangle. Then 63 is the product of two factors one of which is the sum and the other the difference of the sides of the pieces of father and son.

$63=63\times1$ or 21×3 or 9×7.

The side of the greater square, or the father's, is obtained by adding the sum and difference and dividing by two, and of the smaller, or son's, by subtracting the difference from the sum and dividing by two.

This process applied to each of the above pairs of factors gives for the sides of the pieces of father and son, respectively, 32 and 31, 12 and 9, 8 and 1. Since 31 and 9 differ by 23, 32 is the side of Adams's piece, and 9 is the side of Nathan's piece. Also 12 and 1 differ by 11. Therefore 12 is the side of Stoughton's piece, and 1 is the side of Edward's. This leaves 8 for the side of Morris's piece, and 31 for the side of Walter's.

Therefore the boys' names are Edward Morris, Nathan Stoughton, and Walter Adams.

III. **Solution by B. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.**

I. 1. Nathan is not the son of Mr. Adams, since the former has more than 63 square rods of land in excess of the latter.

Therefore either Edward or Walter is the son of Mr. Adams.

2. Edward is not the son of Mr. Stoughton, since the former has more than 63 square rods of land in excess of the latter.

Therefore Nathan or Walter is the son of Mr. Stoughton.

3. Walter is not the son of Mr. Morris, for then would Mr. Adams and Mr. Stoughton each have more land than the other, which is absurd. For then would Edward be the son of Mr. Adams, and Nathan of Mr. Stoughton; but Mr. Adams and Mr. Stoughton have more than 63 square rods, respectively, than Nathan and Edward, and, hence, more than each other.

Therefore either Nathan is the son of Mr. Morris, Walter of Mr. Stoughton, and Edward of Mr. Adams, or Edward is the son of Mr. Morris, Walter of Mr. Adams, and Nathan of Mr. Stoughton.

5. This indefinite solution suggests the idea that the proposer intended as a condition what is neither expressly stated nor necessarily implied in the problem, viz., that all the numbers are integral.

The second solution is on this assumption.

II. 1. Let x stand for the side, in rods, of any square, and $x+y$ for the side of a square whose length is y rods longer. Then $2xy+y^2=63$, whence $x=(63/2y)-\frac{1}{2}y$.

Now, the only positive integral values of y which will make x a positive integer are 1, 3, and 7, the values of x corresponding to which are, respectively, 31, 9, and 1. Hence, the pairs of squares differing by 63 square rods are as follows: (1) Sides, 32 rods and 31 rods; (2) 12 rods and 9 rods; (3) 8 rods and 1 rod.

2. Now, applying the first condition of the problem, it is evident that Mr. Adams owns the largest square, and that his son is Walter, that Nathan is the son of Mr. Stoughton, and Edward of Mr. Morris.

IV. **Solution by SYLVESTER ROBINS, North Branch Depot, N. J.**

$63=63\times1=21\times3=9\times7=32^2-31^2=12^2-9^2=8^2-1^2$.

$32-9=23$.

32^2 belongs to father Adams. 31^2 belongs to Mr. Adams's son.

9^2 belongs to Nathan. 12^2 to Nathan's father.

$12-1=11$.

The 12^2 belongs to father Stoughton. 1^2 belongs to Edward. 9^2 belongs to Nathan Stoughton.

The 8^2 belongs to father Morris. 1^2 to Edward Morris, and the 31^2 to Walter Adams.

Also solved by *COOPER D. SCHMITT*, *J. SCHEFFER*, and *G. B. M. ZERR*.

NOTE. Professor Scheffer sent in solutions of problems 118 and 119, and Professor Ellwood of problem 119. In the published solutions of No. 119, the following errors occur in the statement: For $1.\frac{9}{.100}$, read $1.\frac{9}{.001}$; line 8 from the bottom, for "\$9001, the selling price," read "\$9001, the cost price;" at top of page 270, for $1+\frac{9}{.009}$, read $1+\frac{9}{.001}$.

ALGEBRA.

97. Proposed by F. M. SHIELDS, Coopwood, Miss.

A farmer had 2080 pounds of grain at the depot, and gave a wagoner .75 cents per 100 pounds to haul it, paying him in the *same* grain at the following prices, viz.: 3-10 of the hauling bill was paid in corn at .58 cents per bushel of 56 pounds; 3-5 was paid in wheat at 1.55 cents per bushel of 60 pounds, and the balance of the bill was paid in oats at .36 cents per bushel of 32 pounds, the wagoner not charging for hauling his own grain. The load being delivered, how many bushels of each kind of grain did the wagoner get, and how many bushels of each kind did the farmer have left after paying the wagoner?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and C. B. GOULD, Colorado College, Colorado Springs, Col.

Let x=the hauling bill.

$x\div.75=\frac{4}{3}x$=number of hundredweight hauled.

$\therefore$ $400x/3$=number of pounds hauled.

$(\frac{3}{10}x\div.58)56=\frac{3}{10}x\times\frac{50}{29}\times\frac{56}{1}=840x/29$, pounds corn received by wagoner.

$(\frac{6}{10}x\div1.55)60=\frac{6}{10}x\times\frac{20}{31}\times\frac{60}{1}=720x/31$, pounds wheat received by wagoner.

$(\frac{1}{10}x\div.36)32=\frac{1}{10}x\times\frac{25}{9}\times\frac{32}{1}=80x/9$, pounds oats received by wagoner.

$\therefore$ $2080-400x/3=840x/29+720x/31+80x/9$.

$\therefore$ x=\$10.698843, hauling bill.

$3x/5.80=5.5338843$ bushels of corn wagoner received.

$6x/15.50=4.1414877$ bushels of wheat wagoner received.

$x/3.60=2.9719008$ bushels of oats wagoner received.

$400x/3=1426.5124$ pounds hauled for farmer.

$\frac{3}{10}$ of $400x/3=40x$, $\frac{6}{10}$ of $400x/3=80x$, $\frac{1}{10}$ of $400x/3=40x/3$.

$40x\div56=7.6420307$ bushels of corn farmer had left.

$80x\div60=14.2651240$ bushels of wheat farmer had left.

$40x/3\div32=4.45785125$ bushels oats farmer had left.

II. Solution by J. D. CRAIG, Frankfort, N. J., and P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

Let $40x$=number of pounds of grain farmer retained.

Then $\frac{3}{4}\times40x=30x$=cents paid for carting.

$\frac{3}{10}$ of $30x=9x$=cents paid for carting corn.

$\frac{3}{5}$ of $30x=18x$=cents paid for carting wheat.

$\frac{1}{10}$ of $30x=3x$=cents paid for carting oats.

$9x\div\frac{3}{4}=12x$=pounds of corn farmer retained.

$18x\div\frac{3}{4}=24x$=pounds of wheat farmer retained.

$3x \div \frac{3}{4} = 4x =$ pounds of oats farmer retained.

$9x/58 \times 56 = \frac{252}{29}x =$ pounds of corn wagoner received.

$18x/155 \times 60 = \frac{216}{31}x =$ pounds of wheat wagoner received.

$3x/36 \times 32 = \frac{8}{3}x =$ pounds of oats wagoner received.

$40x + \frac{252}{29}x + \frac{216}{31}x + \frac{8}{3}x = 2080$ pounds of grain..................... (1).

Solving, $x = 35\frac{401}{605}$.

Therefore, $12x = 427\frac{577}{605}$ pounds of corn retained by farmer.

$24x = 855\frac{549}{605} =$ pounds of wheat retained by farmer.

$4x = 142\frac{394}{605} =$ pounds of oats retained by farmer.

$\frac{252}{29}x = 309\frac{543}{605} =$ pounds of corn paid wagoner.

$\frac{216}{31}x = 248\frac{296}{605} =$ pounds of wheat paid wagoner.

$\frac{8}{3}x = 95\frac{61}{605} =$ pounds of oats paid wagoner.

Reducing to bushels we find that the farmer retains $7\frac{1469}{4235}$ bushels corn, $14\frac{802}{3025}$ bushels of wheat, and $4\frac{277}{605}$ bushels of oats.

The wagoner gets $5\frac{6019}{16940}$ bushels corn, $4\frac{428}{3025}$ bushels wheat, and $2\frac{588}{605}$ bushels oats.

III. **Solution by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.; ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Ia.; and O. S. WESTCOTT, Principal North Division High School, Chicago, Ill.**

Let x, y, and z equal, respectively, the number of pounds of corn, wheat, and oats that the farmer had left. Then from the condition of problem we have

$$\frac{3}{10}\left[\frac{.75}{.58}\,\frac{(x+y+z)}{100}\right]56 = \frac{x}{100} \times \frac{.75}{.58} \times 56,$$

or, by reducing, we have $7x - 3y + 3z = 0$......(1), and in like manner we get $3x - 2y + 3z = 0$......(2), and $x + y - 9z = 0$ .....(3), from which we get $y = 2x = 6z$.

$\therefore\ x + y + z = 10z$.

$.75 . \dfrac{z}{100} . \dfrac{32}{.36} = \frac{2}{3}z =$ pounds of oats wagoner received, and in like manner $\frac{54}{31}z$ and $\frac{63}{29}z$ equal pounds of wheat and corn that the wagoner received for hauling.

$\therefore$ we have $2080 - 10z = \frac{63}{29}z + \frac{54}{31}z + \frac{2}{3}z$.

$\therefore\ z = 142\frac{5122}{7865}$ pounds or $4\frac{3601}{7865}$ bushels of oats farmer received.

$y = 6z = 855\frac{7137}{7865}$ pounds or $14\frac{62561}{235950}$ bushels of wheat farmer received.

$x = 3z = 427\frac{7501}{7865}$ pounds or $7\frac{35347}{55055}$ bushels of corn farmer received.

$\therefore$ If the wagoner received 75 cents per 100 pounds for hauling, he received—

$\frac{747968}{7865}$ pounds or $2\frac{7644}{7965}$ bushels of oats,

$\frac{2437344}{7865}$ pounds or $5\frac{4199}{7865}$ bushels of corn,

$\frac{1954368}{7865}$ pounds or $4\frac{11128}{78650}$ bushels of wheat.

We understand the problem to mean that the wagoner received corn for hauling corn, wheat for wheat, etc.

CALCULUS.

NOTE ON PROBLEM 84, BY DR. E. WOELFFING, STUTTGART, GERMANY.

The solution of question 84 can be found in some theorems proved by W. Merkelbach (*Ueber Rollkurven welche von einer Graden eingehüllt werden*, Diss. Marburg, 1881). The first of them is the following:

If a curve, C, rolls upon another curve, C', and a point, P, in the plane of C describes a straight line L, and we make afterwards the curve C' roll upon the curve C, then a straight line, L, in the plane of C' will always pass through P (page 18 of the paper quoted).

Now (and this is the second theorem of Merkelbach) if a sinusoid rolls upon an ellipse, a straight line in the plane of the former passes through a focus of the latter (page 24); therefore, if the ellipse rolls upon the sinusoid, any one of the foci of the former will describe a straight line.

Stuttgart, Germany, July 19, 1899.

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91. Proposed by GUY B. COLLIER, Schenectady, N. Y.

Find the area of a loop of the curve $r^2\cos\theta=a^2\sin3\theta$. [From Hall's *Differential and Integral Calculus*].

I. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; M. C. STEVENS, A. M., Professor of Mathematics, Purdue University, Lafayette, Ind.; WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; ELMER SCHUYLER, Reading, Penna.; and J. SCHEFFER, A. M., Hagerstown, Md.

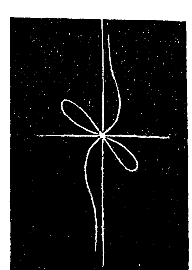
The curve has two equal loops, one in the first and the other in the third quadrant.

The limits of θ are 0 and $\frac{1}{3}\pi$.

$$\therefore\ A=\tfrac{1}{2}\int_0^{\frac{1}{3}\pi} r^2\,d\theta=\tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi}\frac{\sin3\theta\,d\theta}{\cos\theta}$$

$$=\tfrac{1}{2}a^2\int_0^{\frac{1}{3}\pi}(4\sin\theta\cos\theta-\tan\theta)d\theta=\tfrac{1}{4}a^2(3-2\log2).$$

II. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and GEORGE LILLEY, Ph. D., Professor of Mathematics, University of Oregon, Eugene, Ore.



The shape of the curve is seen in the diagram.

The limits are evidently from 0° to 30° for a loop.

$$\text{Then } A=\tfrac{1}{2}\int r^2\,d\theta=\tfrac{1}{2}\int_0^{\frac{1}{6}\pi}\frac{a^2\sin3\theta}{\cos\theta}d\theta$$

$$=\frac{a^2}{2}\int_0^{\frac{1}{6}\pi}\frac{3\sin\theta-4\sin^3\theta}{\cos\theta}d\theta=\frac{a^2}{2}\int_0^{\frac{1}{6}\pi}3\tan\theta\,d\theta$$

$$-2a^2\int_0^{\frac{1}{6}\pi}(\tan\theta-\sin\theta\cos\theta)d\theta$$

$$=\frac{3a^2}{2}\log\sec\theta-2a^2\log\sec\theta+a^2\sin^2\theta\Big]_0^{\frac{1}{6}\pi}=a^2\sin^2\theta-\frac{a^2}{2}\log\sec\theta\Big]_0^{\frac{1}{6}\pi}$$

$$=\frac{3a^2}{4}-\frac{a^2}{2}\log 2.$$

[NOTE. In the figure, the loop in the fourth quadrant should be in the first, and the one in the second in the third.

92. Proposed by B. F. SINE, Principal of Capon Bridge Normal School, Capon Bridge, W. Va.

How much wood is taken from a log 12 inches in diameter, by boring a two-inch hole through the center, the axis of the hole being perpendicular to axis of log ?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J SCHEFFER, A. M., Hagerstown, Md., and WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

$V=8\int\int\int dxdydz=$volume.

The equation of the surface of the cylinder corresponding to the log is

$$x^2+y^2=36=R^2,$$

and the equation of the surface of the cylinder corresponding to the auger-hole is

$$x^2+z^2=1=r^2.$$

$$V=8\int\int\int ydxdz=8\int_0^r\int_0^{\sqrt{(r^2-x^2)}}\sqrt{(R^2-x^2)}dxdy.$$

$$\therefore\ V=8\int_0^r\sqrt{[(r^2-x^2)(R^2-x^2)]}dx$$

$$=\tfrac{8}{3}R^3\{[1+(r/R)^2]E(r/R)-[1-(r/R)^2]F(r/R)\}$$

$$=576\{(1+\tfrac{1}{36})E(\tfrac{1}{6})-(1-\tfrac{1}{36})F(\tfrac{1}{6})\}$$

$$=576\{\tfrac{37}{36}E(\tfrac{1}{6})-\tfrac{35}{36}F(\tfrac{1}{6})\}\text{ cubic inches.}$$

Also $V=8\int_0^r\sqrt{(r^2-x^2)}\Big[R-\frac{x^2}{2R}-\frac{x^4}{8R^3}-\ldots\ldots-\frac{1.3.5.7\ldots.(2n-1)x^{2n}}{2.4.6\ldots.2nR^{2n-1}}\Big]dx.$

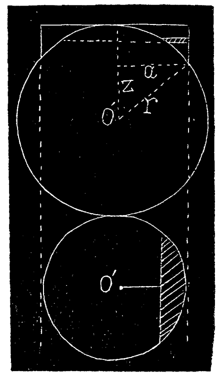
Now $\int_0^r\sqrt{(r^2-x^2)}x^{2n}dx=\frac{1.3.5.7\ldots.(2n-1)}{2.4.6.8\ldots.(2n+2)}\pi r^{n+2}.$

$$\therefore\ V=4r^2R\pi\Big[\tfrac{1}{2}-\frac{r^2}{16R^2}-\frac{r^4}{128R^4}-\ldots\ldots-\frac{(1.3.5.7\ldots\ 2n-3)^2(2n-1)r^{2n}}{(2.4.6\ldots\ldots\ 2n)^2(2n+2)R^{2n}}-\ldots\Big]$$

$$\therefore\ V=24\pi(\tfrac{1}{2}-\tfrac{1}{576}-\tfrac{1}{165888}-\tfrac{5}{95551488}-\ldots.-)=37.56784\text{ cubic inches.}$$

II. Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

This is easily seen to depend on the finding of the volume enclosed between the larger cylinder and a tangent plane, and the smaller cylinder.



Let r=radius of large cylinder.

a=radius of small cylinder.

z=distance of cutting plane from axis of the large cylinder.

We have then to evaluate the integrals

$$a^2\int_{\sqrt{(r^2-a^2)}}^{r}\cos^{-1}\frac{\sqrt{(r^2-z^2)}}{a}dz-\int_{\sqrt{(r^2-a^2)}}^{r}(r^2-z^2)\sqrt{(a^2-r^2+z^2)}dz.$$

Putting $\dfrac{\sqrt{(r^2-z^2)}}{a}=\sin\theta$, the first integral becomes

$$-\frac{\pi a^4}{2}\int\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}+a^4\int\frac{\theta\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}.$$

The second becomes $-a^4\displaystyle\int\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}$.

$$\int\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}=-\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}.$$

Combining the second integral with the latter part of the first, we have

$$\int(\theta+\sin\theta\cos\theta)\frac{\sin\theta\cos\theta d\theta}{\sqrt{(r^2-a^2\sin^2\theta)}}$$

which integrated by parts gives

$$\frac{-(\theta+\sin\theta\cos\theta)\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}+\int\frac{2\cos^2\theta\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta.$$

The last integral

$$=2\int\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta-2\int\sin\theta\frac{\sqrt{(r^2-a^2\sin^2\theta)}}{a^2}d\theta.$$

We have then

$$\tfrac{1}{2}\text{ Volume}=\left[\frac{\pi a^2}{2}\sqrt{(r^2-a^2\sin^2\theta)}-a^2(\theta+\sin\theta\cos\theta)\sqrt{(r^2-a^2\sin^2\theta)}\right]_0^{\frac{1}{2}\pi}$$

$$+2a^2\int\sqrt{(r^2-a^2\sin^2\theta)}d\theta-2a^2\int_0^{\frac{1}{2}\pi}\sin^2\theta\sqrt{(r^2-a^2\sin^2\theta)}d\theta.$$

When $\theta=0$, the first line is $\frac{1}{2}(\pi a^2 r)$.
When $\theta=\frac{1}{2}\pi$, its value is 0.

To evaluate $\int\sqrt{(r^2-a^2\sin^2\theta)}d\theta$, put $a=re$, expand by binomial formula, and integrate the terms.

$$(1-e^2\sin^2\theta)^{\frac{1}{2}}=1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\tfrac{1}{16}e^6\sin^6\theta-\ldots\ldots$$

Then $\displaystyle\int_0^{\frac{1}{2}\pi}\sqrt{(r^2-a^2\sin^2\theta)}d\theta=\tfrac{1}{2}(\pi r)(1-\tfrac{1}{4}e^2-\tfrac{3}{64}e^4-\tfrac{5}{256}e^6\ldots\ldots)$

The remaining integral treated in the same way gives

$$\tfrac{1}{2}\pi r(\tfrac{1}{2}-\tfrac{3}{16}e^2-\tfrac{5}{128}e^4-\tfrac{35}{2048}e^6-\ldots\ldots$$

Finally, Volume$\displaystyle=\frac{\pi a^2 r}{8}\left(e^2+\tfrac{1}{8}e^4+\frac{5e^6}{128}+\ \ldots\right)$

In our example, $e=\frac{1}{6}$, $a=1$, $r=6$.

Hence, $\displaystyle V=\frac{9\pi}{2}\left(\frac{1}{36}+\frac{1}{8\times1296}+\frac{5}{128\times6^6}+\ldots\ldots=\tfrac{1}{8}\pi(1+\tfrac{1}{288}\right)$ nearly.

This subtracted from $\pi a^2 r$ and the result doubled gives the volume common to the two cylinders.

MECHANICS.

90. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

Adopting the hypothesis that the planets were originally all one mass revolving about a fixed center and were formed by an explosion of this mass at some point in its path; prove that, if the law of nature were that force varies directly as the distance, the planets would all have collided again simultaneously, and find an expression for the time between the explosion and collision.

Solution by the **PROPOSER**.

Regarding the original mass as a particle, the pieces after explosion, no matter what their initial velocities or directions, would all move in concentric ellipses; and as their paths intersect in one point, viz., the position of the original mass at the moment of explosion, they must all have another point in common at the extremity of the common diameter through the first common point.

We have for the equations of motion for any piece

$$\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=0 \;\ldots\ldots(1). \qquad m\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]=-r\ldots\ldots(2).$$

From (1) by integration,

$$r^2\frac{d\theta}{dt}=h\ldots\ldots(3),$$

a constant depending on the initial velocity and angle of projection.

To integrate (2) let $u=1/r$. Then

$$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}=-h\frac{du}{dt} \text{ from (3).}$$

$$\therefore \frac{d^2r}{dt^2}=-h\frac{d}{dt}\left(\frac{du}{d\theta}\right)=-h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting these results in (2) and using (3) we have

$$\frac{d^2u}{d\theta^2}=-u+\frac{k}{h^2u^3}\ldots\ldots(4),$$

where k is a constant depending on the force of attraction.

Multiply by $2\frac{du}{d\theta}$ and integrate and we have

$$\left(\frac{du}{d\theta}\right)^2=-u^2-\frac{k}{h^2u^2}+c_1\ldots\ldots(5),$$

$$\theta=\int\frac{hudu}{\sqrt{(c_1h^2u^2-h^2u^4-k)}}=\tfrac{1}{2}\cos^{-1}\left(\frac{2hu^2-c_1h}{\sqrt{(c_1^2h_2-4k)}}\right)+c_2.$$

Hence simplifying and restoring value of u we get for the equation of the path

$$\frac{1}{r^2}=\frac{c_1}{2}-\frac{\sqrt{(c_1^2h^2-4k)}}{2h}+\frac{\sqrt{(c_1^2h^2-4k)}}{h}\cos^2(\theta-c_2).$$

Transforming to rectangular coodinates this becomes

$$\frac{x^2}{\dfrac{2h}{c_1h+\sqrt{(c_1^2h^2-4k)}}}+\frac{y^2}{\dfrac{2h}{c_1h-\sqrt{(c_1^2h^2-4k)}}}=1.$$

Let A be area of the ellipse, and a, b, its semi-axes Then

$$A=\pi ab=\pi\times\frac{2h}{c_1h+\sqrt{(c_1^2h_2-4k)}}\times\frac{2h}{c_1h-\sqrt{(c_1^2h_2-4k)}}=\frac{\pi h}{\sqrt{k}}.$$

For the area of any curve, we have from the calculus

$$A=\tfrac{1}{2}\int_{\theta_0}^{\theta_1}r^2\,d\theta.$$

If we regard the angle θ_0 as the initial angle it is 0, and as θ is a function of the time, this integral may be written

$$A=\tfrac{1}{2}\int_0^{t_1}r^2\,d\theta=\tfrac{1}{2}\int_0^t r^2\frac{d\theta}{dt}dt=\tfrac{1}{2}h\int_0^{t_1}dt=\tfrac{1}{2}ht \text{ by } (3).$$

$\therefore$ $t=2A/h$. That is, the periodic time always equals twice the area swept over by the radius vector divided by the constant h.

Hence for the given ellipse we have

$$t=\frac{2\pi h/\sqrt{k}}{h}=\frac{2\pi}{\sqrt{k}}.$$

Therefore the time it would require for a piece to travel from the point of explosion to the next point of intersection is $\tfrac{1}{2}t-\pi/\sqrt{k}$, and as k is a constant the same for all pieces, we see this time is the same for all, and hence they must collide simultaneously.

91. Proposed by CHARLES C. CROSS, Whaleyville, Va.

The bow of a boat which is a inches wide is inclined at an angle α. When in motion in perfectly calm water the water was found to rise b inches on the bow. Required the velocity of the boat.

No solution of this problem has been received.

92. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A particle, starting at the vertex, slides down a smooth parabolic curve. Find the initial velocity of the particle so that it may leave the curve at the extremity of semi-latus rectum.

Solution by GEORGE R. DEAN, A. M., Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Take the vertex as origin and positive x downwards.

The equation of the curve is $y^2=2px$.

Let N be the normal reaction, v the velocity, θ the angle between normal and x-axis, m the mass of particle, ρ the radius of curvature.

Then $N=mv^2/\rho+mg\cos\theta$.

$$\rho=\frac{(y^2+p^2)^{\frac{3}{2}}}{p^2},\quad v^2=v_0^2+2gx,\quad \cos\theta=\frac{p}{\sqrt{(y^2+p^2)}}.$$

Accordingly, $N=mg(p-\frac{v_0^2}{g})=\frac{m}{p}(pg-v^2)$.

If $v_0^2=pg$ the normal reaction is always zero.
If $v_0^2>pg$ the particle leaves the curve at once.
If $v_0^2<pg$ the particle is constrained to move on the curve.
Then the particle either—
1°. Describes the curve freely without leaving it; or,
2°. Leaves the curve at the beginning of the motion; or,
3°. Describes the curve under constraint without leaving it.

Also solved by *G. B. M. ZERR*, and the *PROPOSER*.

DIOPHANTINE ANALYSIS.

75. **Proposed by CHARLES CARROLL CROSS, Whaleyville, Va.**

Arrange the consecutive integers 1 to n^2 as a magic square, where n is odd. Apply when $n=9$.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

This solution for odd magic squares is adapted from a similar solution found in an article on "Evening Entertainments."

We divide a large square into as many little spaces as there are numbers in the magic square.

We conceive a similar large square, similarly divided, bordering on the right, another on the bottom, another on the left, and a fourth meeting the right bottom corner of the given square.

We fill in the numbers consecutively, commencing with unity, *from left diagonally down to right, beginning with the space immediately below the center space.*

When reaching the limit of the given square, we continue the next number into the "bordering square." This number will then be placed in the *corresponding space of the given square.*

When, in filling in the numbers, we meet a space that is already occupied, we take the space *next diagonally down left from this occupied space*, and continue as before.

Take a magic square having 9 numbers on a side. There will then be 9^2, or 81 spaces in the square, to be filled with the numbers 1 to 81 inclusive.

Beginning with the space next below the center space, and filling in diagonally down to right, we find 4 reaches the limit of the square. Whence 5 occupies, in the "bordering square," the right upper corner. We now put 5 in the right upper corner of the given square, and find it at the limit of the square. Whence 6 occupies, in the "bordering square," the space next below the left upper corner. We now place 6 in the corresponding space of the given square, and proceed 6, 7, 8, 9, when we meet the space occupied by 1. We then put 10 in the space next diagonally down left from 1, and proceed 10, 11, 12, etc., until all the spaces are filled.

		37	78	29	70	21	62	13	54	5		
	46	6	38	79	30	71	22	63	14	46	6	
		47	7	39	80	31	72	23	55	15	47	
		16	48	8	40	81	32	64	24	56	16	
		57	17	49	9	41	73	33	65	25	57	
		26	58	18	50	1	42	74	34	66	26	
		67	27	59	10	51	2	43	75	35	67	
		36	68	19	60	11	52	3	44	76	36	
		77	28	69	20	61	12	53	4	45	77	
		37	78	29	70	21	62	13	54	5	46	

AVERAGE AND PROBABILITY.

79. **Proposed by the late ENOCH BEERY SEITZ.**

Two equal spheres touch each other externally. If a point be taken at random within each sphere, show that (1) the chance that the distance between the points is less than the diameter of either sphere is 13/35, and (2) the average distance between them is $11/5r$. [This is problem 5835, *Educational Times*, of London.]

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1) Let A, B be the centers and C the point of contact of the two spheres, each radius r.

From any point P in DC with a radius$=2r$ describe a sphere cutting B in Q, R. From B as a center with a radius BP describe a sphere cutting A in K, M. If P is the first point, the second point must fall within the double-convex lens $CQRC$. P may fall anywhere on the zone KPM and the second point must fall in a section of B equal to the double-convex lens $CQRC$.

From P as center with a radius $PS<2r$ but$>PC$, draw the zone SLT. Let $DP=x$, $PS=y$, area of zone $KPM=2\pi.BP.PG$, area of zone $SLT=2\pi.PS.HL$.

$BP=3r-x$, $AG=r-x-PG$, $BG=3r-x-PG$, $PS=y$, $BH=3r-x-y+HL$, $PH=y-HL$.

$KG^2=r^2-(r-x-PG)^2=(3r-x)^2-(3r-x-PG)^2$.

$\therefore\ PG=x(2r-x)/4r$.

$SH^2=r^2-(3r-x-y+HL)^2=y^2-(y-HL)^2$.

$\therefore\ HL=[r^2-(3r-x-y)^2]/2(3r-x)$.

$\therefore$ Area of zone $KPM=(\pi x/2r)(3r-x)(2r-x)$.

Area of zone $SLT=[\pi y/(3r-x)][r^2-(3r-x-y)^2]$.

Let p=chance, $\triangle$=average distance.

$$\therefore\ p=\{\pi^2/[2r(\tfrac{4}{3}\pi r^3)^2]\}\int_0^{2r}x(3r-x)(2r-x)dx\int_{2r-x}^{2r}[y/(3r-x)][r^2-(3r-x-y)^2]dy$$

$$=(3/128r^7)\int_0^{2r}(14rx^5-x^6-48r^2x^4+48r^3x^3)dx=(3/128r^7)(1664r^7/105)=\tfrac{13}{35}.$$

$$2.\ \triangle=\{\pi^2/[2r(\tfrac{4}{3}\pi r^3)^2]\}\int_0^{2r}x(3r-x)(2r-x)dx\int_{2r-x}^{4r-x}[y^2/(3r-x)][r^2-(3r-x-y)^2]dy$$

$$=(3/40r^4)\int_0^{2r}(92r^3x-106r^2x^2+40rx^3-5x^4)dx=(3/40r^4)(88r^5/3)=11r/5.$$

80. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A box contains 100 balls marked from 1 to 100. 13 balls are drawn at random. What is the chance that the balls marked from 1 to 10 are included in the 13 drawn ?

Solution by J. W. YOUNG, Columbus, Ohio.

Since in all the favorable chances only three balls may vary, the total number of favorable chances is ${}^{90}C_3$, *i. e.*, the number of combinations of 90 things taken 3 at a time.

The total number of ways in which the balls may be drawn is, of course, ${}^{100}C_{13}$.

Hence the desired probability is equal to

$$\frac{{}^{90}C_3}{{}^{100}C_{13}}=\frac{\dfrac{90.89.88}{1.2.3}}{\dfrac{100.99.98.97.96.....89.88}{1.2.3.5......13}}=\frac{1}{67515927540}.$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

124. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand ? When is the hour hand midway between 4 and the minute hand ?

125. **Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.**

A Quaker once, we understand
For his three sons laid off his land,
And made three equal circles meet
So as to bound an acre neat.
Now in the center of the acre,
Was found the dwelling of the Quaker;
In centers of the circles round,
A dwelling for each son was found.
Now can you tell by skill or art
How many rods they live apart?

$_*{}^*{}_*$ Solutions of these problems should be sent to B. F. Finkel not later than March 10.

ALGEBRA.

115. **Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.**

Find the conditions of the coefficients of a general biquadratic equation so that it may be solved by quadratics.

116. **Proposed by ARTEMAS MARTIN, A. M., Ph. D., LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.**

Solve the equations:

$$w(xy+xz+yz)=a;\ x(wy+wz+yz)=b;$$
$$y(wx+wz+xz)=c;\ z(wx+wy+xy)=d.$$

$_*{}^*{}_*$ Solutions of these problems should be sent to J. M. Colaw not later than March 10.

GEOMETRY.

135. **Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.**

If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

136. **Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.**

Construct a triangle having given the base, the median line to the base, and the difference of the base angles.

137. **Proposed by J. W. YOUNG. Fellow and Assistant, Ohio State University, Columbus, O.**

A right cone has its vertex in a horizontal plane, its axis being perpendicular to the plane. A string has one extremity attached to a point on the cone. The other extremity, P, of the string is kept in the plane, and the string is then wound around the cone, without being allowed to slip. Show that the spiral generated by P cuts all straight lines through the vertex at the same angle.

$_*{}^*{}_*$ Solutions of these problems should be sent to B. F. Finkel not later than March 10.

CALCULUS.

106. **Proposed by M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind.**

$$\int_0^\pi \frac{\cos rx dx}{1-2a\cos x+a^2}=\frac{\pi r^2}{1-a^2}.$$

[*Williamson's Integral Calculus*, 6th Edition, page 174.]

107. Proposed by **F. P. MATZ, M. Sc., Ph. D.,** Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

The speed of signaling in submarine telegraph-cable varies as $x^2\log(1/x)$, in which x is the ratio of the radius of the core to that of the covering. Prove that the *maximum speed* is attained when this ratio is $1:\sqrt{e}$.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

MECHANICS.

101. Proposed by **ALOIS F. KOVARIK,** Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

Find the center of gravity of a cone that has a specific gravity of 1 (one) at the top and 2 (two) at the base.

102. Proposed by **WALTER H. DRANE,** Graduate Student, Harvard University, Cambridge, Mass.

A heavy particle with a light string attached is placed on the edge of a smooth table. A boy, holding the string horizontally, runs at right angles to the string. Determine the motion of the particle (1) when the boy runs with a uniform velocity; (2) when he runs with a uniform acceleration.

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

DIOPHANTINE ANALYSIS.

83. Proposed by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find three numbers in arithemetical progression whose sum is a square and cube.

84. Proposed by **SYLVESTER ROBINS,** North Branch Depot, N. J.

The nth term of an infinite series of "nests" contains all the prime, integral, rational parallelopipeds that have 3^n for their solid diagonals. It is required to determine the general expression for N=the number of such solids in nth term, and to exhibit the dimensions of all the "eggs" in the first six nests.

*** Solutions of these problems should be sent to J. M. Colaw not later than March 10.

AVERAGE AND PROBABILITY.

88. Proposed by **G. B. M. ZERR, A. M., Ph. D.,** Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

89. Proposed by **B. F. FINKEL, A.M., M. Sc.,** Professor of Mathematics and Physics, Drury College, Springfield, Mo.

An inch auger-hole is bored at random through a six-inch sphere. Find the average volume of the auger-hole.

90. Proposed by **WALTER H. DRANE,** Graduate Student, Harvard University, Cambridge, Mass.

During a heavy rain storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond? [From *Byerly's Integral Calculus.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than March 10.

MISCELLANEOUS.

85. **Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knox ville, Tenn.**

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

86. **Proposed by GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.**

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the payment on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

87. **Proposed by A. H. HOLMES, Brunswick, Me.**

Find $f(x)$ from $f(x+1)-f^2(x)=x$.

NOTES.

Dr. E. M. Blake, formerly instructor in mathematics in Purdue University, has been appointed as Honorary Fellow in Mathematics in Cornell University.

The two books, Whitehead's Universal Algebra and Killing's Einführung in die Grundlagen der Geometrie, which were particularly signalized in Halsted's Report on Progress in Non-Euclidean Geometry, have been entered in competition for the Lobachevski Prize of 1900.

On the twenty-fourth of December, 1899, the Physico-Mathematic Society of Kazan (Russia) celebrated a Jubilee in honor of the twenty-fifth year of professional and scientific service of its President, Professor A. Vasiliev. It is also the fifteenth year of his presidency. Professor Vasiliev has been an extraordinarily important figure in Russian science. Outside of Russia he has chiefly been known for his remarkable discourse on Lobachevski, Englished by Halsted. A German translation of his book on Tchebychev is to be published this month by Teubner at Leipzig. The first volume of an edition of Tchebychev's Collected Works, in French, has just appeared, edited by the academicians Markof and Sonine. It contains a very fine portrait of the great mathematician and the first thirty-four Memoirs of Vasiliev's list.

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from December Number.]

CHAPTER V.

RELATIVE MULTIPLICATION.

56. DEFINITION.—In the preceding chapter no reference was made to the space in which the factors multiplied were contained. Now, in ordinary multiplication of geometrical magnitudes, there is a limit beyond which one can not go. For instance, when one has multiplied the length, breadth, and thickness together, he can add no other dimension. This suggests the idea of taking any arbitrary number as n for the number of dimensions of the space considered. In any investigation, then, what we will call "the space considered" is that space of the original units which contains all the quantities involved. Multiplications made with reference to the space considered are called *Relative Multiplications.*

57. DEFINITION.—If in a space of the nth order, the combinatory product of the original units $e_1 e_2 \;\ldots\ldots e_n$, is set equal to the scalar unity, and E is a unit of any order, (*i. e.* either one of the original units or a combinatory product of two or more of them), then the *complement of* E is $+E'$ or $-E'$, where E' is the combinatory product of all the units which do not appear in E. The complement of E is $+E'$ when $[EE']=+1$; and $-E'$ when $[EE']=-1$. Let the com-

plement of E be denoted by $|E$. This mark, the sign of the complement, in Grassmann is a vertical line somewhat longer that the caps and about as heavy as the vertical stroke in cap N. Then

$$|E=[EE']E'.$$

The end sought is to get $[E|E]=+1$. To show that this is attained, multiply the equation above through by E. Then

$$[E\,|\,E]=[EE'][EE']=+1,$$

since whether $[EE']=+1$ or -1, $[EE'][EE']=+1$.

The reason why we have this ambiguity of sign in the product $[EE']$ is because each original unit is allowed to have either the plus or minus sign.

In particular we have

$$|1=1, \text{ or } |\alpha=\alpha,$$

by multiplying the first equation through by α.

58. Definition.—By the *Complement* of any quantity A will be understood that quantity $|A$ which is obtained by replacing each product of units in the derived expression for A by its complement. Expressing the same in a formula we have

$$|A\equiv|(\alpha E_1+\alpha_2 E_2+\ldots\ldots)=\alpha_1\,|E_1+\alpha_2\,|E_2+\ldots\ldots$$

where E_1, E_2, ... are products of units of any order.

59. *The complement of the complement of any quantity A is equal either to A, or to $-A$, according as $(-1)^{qr}$ is $+$ or $-$*, where q and r are the orders of the quantity and its complement.

The Proof depends on 41.

60. *If the order of a space n is odd. $||A=A$: if n is even. $||A=(-1)^qA$, where q is the order of A.*

Proof.—By 59, $||A=(-1)^{q(n-q)}A$. Then if n is odd, $q(n-q)$ is even, whether q be even or odd : but if n is even, then if q is even $q(n-q)$ is even, and if q is odd $q(n-q)$ is odd. The theorem as stated readily follows.

61. Definition.—If the sum of the orders of two units is less than or equal to the order n of the space considered, then by their *progressive product* is understood their outer product (52), with the provision, however, that the product of the n original units is unity. On the other hand if the sum of the orders of two units is greater than the order n of the space, then by their *regressive product* is understood that quantity whose complement is the progressive product of the complements of these units.

Thus, if the sum of the orders of E and $F>n$, we have that

$$|[EF]=[|E|F],$$

where $[e_1 e_2 \ldots\ldots e_n]=1$.

The regressive products can be made plainer by an example. Let 5 be the order of the space considered, $[e_1 e_2 \ldots\ldots e_5]=1$, and let the product of $E_1=[e_1 e_2 e_3 e_4]$ and $E_2=[e_1 e_2 e_5]$ be required.

Changing the order of the factors of E_2, we write $E_2=[e_5 e_1 e_2]$ (37), there being two interchanges. Then

$$[E_1 E_2]=[e_1 e_2 e_3 e_4][e_5 e_1 e_2]=[e_1 e_2 e_3 e_4 e_5 e_1 e_2] \text{ (52)}=[e_1 e_2] \text{ (Rem. Art. 13).}$$

Thus the product of E_1 and E_2 is the product of their common factors e_1 and e_2. The question arises, how can the common factors be selected and their product formed out of the two given factors?

Since E_1 and E_2 together contain all five of the units and neiter $|E_1$ nor $|E_2$ contains e_1 or e_2, the product of $|E_1$ and $|E_2$ contains all the units except e_1 and e_2. Then $|[E_1 E_2]$ contains the factors of $[|E_1|E_2]$. Hence the definition of a regressive product.

62. *If q and r are the orders of A and B and n that of the space considered, the order of the product $[AB]$ is equal to $q+r$ when $q+r<n$, but is equal to $q+r-n$ when $q+r\geqq n$. In the language of the Theory of Numbers if p is the order of the product*

$$p \equiv q+r \text{ (Modulus } n).$$

The proof of this theorem follows the lines of the example of the preceding article.

63. *Similarly, for a larger number of factors,*

$$p \equiv q+r+s+t \ldots\ldots \text{ (Modulus } n).$$

64. *The product of the complements of two quantities is the complement of the product of those quantities, that is to say,*

$$[|A\,|B]=|[AB].$$

Proof.—1. Suppose at first that the sum of the orders α and β of A and B is greater that n, that of the space under consideration. Let $A=\Sigma\alpha_r E_r$, $B=\Sigma\beta_s F_s$, where E_r and F_s are units. Then $|A=\Sigma\alpha_r|E_r$ and $|B=\Sigma\beta_s|F_s$ (58). Thus, we have,

$$[|A\,|B]\equiv[\Sigma\alpha_r|E_r.\Sigma\beta_s|F_s]=\Sigma\alpha_r\beta_s[|E_r|F_s \ldots\ldots\ldots\ldots(28)$$

$$=\Sigma\alpha_r\beta_s|[E_r F_s] \ \ldots\ldots(61)=|\Sigma\alpha_r\beta_s[E_r F_s] \ldots\ldots(58)$$

$$=|[\Sigma\alpha_r E_r.\Sigma\beta_s F_s]\ldots\ldots(28)\equiv|[AB].$$

2. Suppose $\alpha+\beta=n$. Let E and F be products of the original units. Two cases may be distinguished. First, when E and F contain a common factor e_1. It is plain that in this case both $[EF]$ and $[|E|F]$ contain common factors, so that they are each equal to zero (43). Second, when $[EF]=1$. Replacing $|E$ and $|F$ by their values from 57 and noting that $[FE][FE]=+1$, we get $[|E|F]=[EF]$. But as $[EF]=1$, $|[EF]=1$ (57). Thus the law holds for units. Then reasoning as in 1, above, it holds for any quantities.

3. Suppose $\alpha+\beta<n$. The proof of this case is based on 1. of this article by letting $A=|A'$, $B=|B'$, and writing

$$|[A'B']=[|A'\,|B']=[AB].$$

65. *The product of the complements of several quantities is the complement of the product of these quantities.*

Proved by mathematical induction from 64.

66. *The complement of a polynomial is the sum of the complements of its parts.*

Proof from 58.

67. *If E, F, G are units the sum of whose orders is n (the order of the space considered),*

$$[EF.EG]=[EFG]E.$$

PROOF.—We distinguish two cases: either $[EFG]$ contains equal factors or it does not. If it does, then at least one of the units, say e_1, is missing in it. Now let $[EF]=|Q$; then by 57 Q must contain e_1; likewise, let $[EG]=|R$; then R contains e_1. Then $[QR]=0$ (43). Now we have

$$[EF.EG]=[|Q|R]=|[QR]\ (64)=|0=0\ (57).$$

But $[EFG]$ also equals zero since it contains equal factors (43). Thus in this case

$$[EF.EG]=[EFG]E.$$

If $[EFG]$ does not contain equal factors, then it contains all of the original units and no others. Then by 57 and 55,

$$|G=[GEF][EF],\quad |F=[FEG][EG].$$

Since $[GEF]$ and $[FEG]$ are equal to either $+1$ or -1, they can appear on either side of their equations. Hence we can write

$$[EF]=[GEF]\,|G,\quad [EG]=[FEG]\,|F.\quad \text{Whence}$$

$$[EF.EG]=[GEF][FEG][|G|F]=[GEF][FEG]\,|[GF\ldots\ldots\ldots(64)$$

$$=[GEF][FEG][GFE]E\ldots(57)=[GEF][EFG][GEF]E\ldots(41)$$

$$=[EFG]E,$$

since $[GEF][GEF]$, as in (57), equals $+1$.

68. *If A, B, C are simple quantities the sum of whose orders equals n, the order of the space of these quantities.*

$$[AB.AC]=[ABC]A.$$

The proof of this formula (based on (67) on account of its length, is omitted, as are also those of the next four formulas given below.

69.—71. *If A, B, C are simple quantities whose product is of the* 0*th order.*

69. $$[AB.AC]=[ABC]A.$$

70. $$[AB.BC]=[ABC]B.$$

71. $$[AC.BC]=[ABC]C.$$

72. *If A, B, C are simple quantities and the sum of the orders of A and C equals the order of the space considered and B is subordinate* (18) *to A, then*

$$[A.BC]=[AC]B \text{ and } [CB.A]=[CA]B.$$

Remark.—It seems proper to state here that the matter contained in Chapters II—V is taken direct from Grassmann's Ausdehnungslehre of 1862. What the writer has done has been to cut out everything which was not essential to the development of the main principles of the work. What to insert and what to omit constitutes the chief difficulty. In the following chapters (except Chapter VIII) we shall not follow Grassmann very closely.

MATHEMATICAL INDUCTION.

By ARTHUR L. BAKER, C. E., Ph. D., University of Rochester, Rochester, N. Y.

There is such a general lack of presentation of the governing principles of mathematical induction in the text books which refer to the subject, and failure to give a working rule for its application, that it is thought a presentation of such a working rule specifically stated and not left to inference from examples merely, would be acceptable.

Mathematical Induction (not entering into the question of the appropriateness of the name) is a method of proof used where a primary operation and a

secondary law have produced the same series of results to see if the supposed secondary law is valid.

It may be conveniently divided into two cases :

I. When the primary operation is an algebraic one and the secondary law a functional one.

II. When the primary oporation is a functional one and the secondary law an algebraic one.

The necessity for a proof that the secondary law is valid is seen when we find that many apparent secondary laws which hold for a few terms fail later on, in other words, that because an apparent law holds for ten, twenty or thirty terms of a series, we are not at all justified in inferring that it will hold throughout.

A few examples will illustrate this :

x^2+x+17 is prime for sixteen values of x, beginning with $x=0$, but the law fails for the seventeenth term of the series.

x^2+x+41 is likewise prime for forty terms, but the law fails on the forty-first.

The sum of the divisors of $n\,!$ is $\frac{1}{2}(n+1)!$ for a few terms, after which the law fails. And so on for many apparent secondary laws.

I. When the primary operation is a simple algebraic one, and the supposed secondary law is a more complex or functional one.

In this case Mathematical Induction consists in applying the primary operation to the general result produced by the secondary law to see if the next term of the series thus produced by the primary operation is the same as would result from an application of the secondary law. If it is, and the operand was correct, the secondary law is correct.

Example.—$1^3+2^3=9(2-1)^2$, $1^3+2^3+3^3=9(3-1)^2$, giving the apparent secondary law $1^3+2^3+\ \dots\ n^3=9(n-1)^2$.

To test this, add another term (primary operation) giving

$$1^3+2^3+\dots\dots(n+1)^3=9(n-1)^2+(n+1)^3\pm 9n^2,$$

and therefore the secondary law is not correct.

Example.—$1^3+2^3=\left(2\frac{2+1}{2}\right)^2$, $1^3+2^3+3^3=\left(3\frac{3+1}{2}\right)^2$, giving the apparent secondary law,

$$1^3+2^3+\dots\dots n^3=\left(n\frac{n+1}{2}\right)^2.$$

Apply the primary operation by adding another term, giving

$$1^3+2^3+\dots\dots(n+1)^3=\left(n\frac{n+1}{2}\right)^2+(n+1)^3=\left[\ (n+1)\ \frac{n+2}{2}\right]^2$$

and the secondary law is correct.

II. When the primary operation is a functional one and the secondary law is a simple algebraic one.

In this case mathematical induction consists in stating the primary functional operation for two general terms in succeession, and taking any convenient multiple of the first from a convenient multiple of the second. If this difference conforms to the supposed secondary law, the secondary law is true provided the first functional result is true.

Thus let

$$f(x)=F\phi$$

be a term of the series, where x is the element whose change produces the term of the series, and ϕ is the expression which enters as an element of the secondary law. If now

$$A.f(x+1)-B.f(x)=F_1\phi$$

where F_1 has the same algebraic form as F, except as to the values of the constants, then

$$f(x+1)=F_2\phi$$

where F_2 has the same algebraic form as F and F_1 except as to the value of the constants. Hence the form of the function F being persistent for two terms, the law is true if true for F.

A special case is where $F\phi=C.\phi$, C being a mere factor.

Example. — $x^3-x=\binom{6\alpha}{24\beta}$ according as x is $\binom{\text{even}}{\text{odd}}$, α and β being factors, holds for some cases.

1°. $x=2n+1$.

$f(x)=(2n+1)^3-(2n+1)=8n^3+12n^2+4n.$

$f(x+1)=(2n+3)^3-(2n+3)=8n^3+36n^2+52n+24.$

$f(x+1)-f(x)=24n^2+48n+24=24\beta.$ Q. E. I.

2°. $x=2n$.

$f(x)=(2n)^3-2n=8n^3-2n.$

$f(x+1)=(2n+2)^3-(2n+2)=8n^3+24n^2+22n+6.$

$f(x+1)-f(x)=24n^2+24n+6=6\alpha.$ Q. E. I.

Example.—$f(x)=5^{2x+2}-24x-25=576\alpha$ holds for $x=1, 2, 3.$

$f(x+1)=5^{2x+4}-24(x+1)-25.$

$f(x+1)-25f(x)=25(24x+25)-24x-49=576(x+1)=576\alpha.$ Q. E. I.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

122. Proposed by G. B. M. ZERR, A.M., Ph.D.. Professor of Mathematics and Science, Chester High School, Chester, Pa.

Suppose 10% traction stock is 20% better in the market than 5% mining stock; if my income be \$500 from each, how much money have I paid for each, the whole investment bringing $6\frac{2}{3}$%?

Solution by BENJAMIN F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, Ohio; M. A. GRABER, Student in Heidelberg University, Tiffin, Ohio; E. L. SHERWOOD, Professor of Mathematics, Beaver College, Beaver, Pa.; and the PROPOSER.

$\$500 \div .10 = \$5000 =$ par value of traction stock;

$\$500 \div .05 = \$10000 =$ par value of mining stock;

$\$1000 \div .06\frac{2}{3} = \$15000 =$ whole investment.

$\$5000 = \frac{1}{2}$ of \$10000 or the face of the traction stock is one-half the face of the mining stock$=50\%$.

$\therefore\ 100\% = 50\%$ or $1\% = \frac{1}{2}\%$.

$20\% = 20 \times \frac{1}{2}\% = 10\% =$ excess of traction stock over same amount of mining stock.

$50\% + 10\% = 60\% =$ investment in traction stock.

$100\% + 60\% = 160\% =$ whole investment.

$\$15000 \div 1.60 = \$9375 =$ amount invested in mining stock.

$\$15000 - \$9375 = \$5625 =$ amount invested in traction stock.

Also solved, with different results, by *ELMER SCHUYLER*, *ALOIS F. KOVARIK*, *COOPER D. SCHMITT*, and *LESLIE L. LOCKE*.

123. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If $m=2$ cents be the interest on $M=100$ cents for $p=40$ days, find the yearly rate per cent.

Solution by COOPER D. SCHMITT, A. M.. Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa; LESLIE L. LOCKE, Professor of Mathematics, Fredonia Institute, Fredonia, Pa.; ELMER SCHUYLER, Professor of German, Reading, Pa.; and G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

If the interest on M cents for p days is m cents, the interest on M cents for 1 day is m/p cents, and for one year is $360m/p$ cents. Then the annual rate is $\frac{360m/p}{M}$ of 100% which equals $\frac{36000m}{pM}$. In the present example $m=2$, $p=40$, $M=100$. Whence rate$=\frac{72000}{4000}=18\%$.

ALGEBRA.

98. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

A and B agreed to reap a field of grain for 90 shillings. A could reap it in 9 days, and they promised to complete it in 5 days; but B, who did not work as quickly ae he expected, was obliged to call to his assistance C, an inferior workman, who worked the last two days, in consequence of which B received 3*s*. 6*d*. less than would otherwise have been due him. In what time could B and C each reap the field? From *Milne's High School Algebra.*

I. Solution by O. S. WESTCOTT, Principal of North Division High School, Chicago, Ill.

Let x=time in which B could reap the field.

y=time in which C could reap the field.

Then $1/x+\frac{1}{9}=\dfrac{x+9}{9x}$=what A and B can do in one day.

$\therefore \dfrac{9x}{x+9}$=time in which A and B could do the work.

Now $90/x$ represents B's daily wages.

Therefore, $\dfrac{90}{x}\left(\dfrac{9x}{x+9}\right)$=what B should have received if A and B had together completed the job.

But $90/x\times 5=\dfrac{450}{x}$ represents what B did receive.

Hence $\dfrac{90}{x}\left(\dfrac{9x}{x+9}\right)-\dfrac{450}{x}=3\frac{3}{4}$.

From which we obtain $x^2-87x=-1080$, and

$$x=\tfrac{87}{2}\pm\sqrt{(-1080+\tfrac{7569}{4})},\quad x=\tfrac{1}{2}(87\pm 57)=15 \text{ or } 72,$$

and since $\frac{5}{9}+5/x+2/y=1$, we have $\frac{5}{9}+\frac{5}{15}+2/y=1$ and $y=18$, or $\frac{5}{9}+\frac{5}{72}+2/y=1$ and $y=5\frac{1}{3}$.

But it is explicitly stated in the problem that C was the inferior workman; therefore the values $x=15$ and $y=18$ are the values sought.

Similar solutions were received from *COOPER D. SCHMITT*, *ALOIS F. KOVARIK*, *J. SCHEFFER*, *G. W. NASH*, and *J. M. BANDY*.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In five days A does $\frac{5}{9}$ of the work and receives 50 shillings. B receives $40-3\frac{3}{4}=36\frac{1}{4}$ shillings for 5 days work, or $14\frac{1}{2}$ shillings for 2 days work.

$14\frac{1}{2}\div 3\frac{3}{4}=\frac{29}{2}\times\frac{4}{15}=\frac{58}{15}$ times as many days for C to do the work as B.

Let x=number of days it takes B.

Then $58x/15$=number of days it takes C.

$\therefore 5/x+30/58x=\frac{4}{9}$.

$\therefore x=12\frac{12}{29}$ days, the time it takes B.

$58x/15=48$ days, the time it takes C.

Solved in like manner by *H. C. WHITAKER*, *P. S. BERG*, *B. F. SINE*, *J. F. TRAVIS*, and *P. H. PHILBRICK*.

NOTE.—This problem was proposed because we held that the wording of the problem was not sufficiently explicit. The fact that our contributors have taken the different views indicated by the above solutions confirms our opinion. The problem has been worded a little different in the last edition of Dr. Milne's Algebra. The wording of the problem is intended to lead to the first solution. B. F. F.

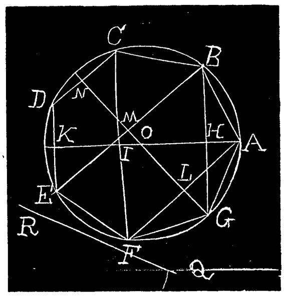
GEOMETRY.

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry*.]

I. Solution by HENRY HEATON, B. Sc., Atlantic, Ia.

Let $ABCDEFG$ represent a regular heptagon circumscribed by the circle whose center is O. Draw a diameter through A. It will be perpendicular to BG, CF, and DE, and it will bisect them in H, I, and K.



Draw any line RQ and for convenience call the perpendiculars from A, B, C, etc., upon RQ, a, b, c, etc. Then $b+g=2h$, $c+f=2i$, and $d+e=2k$.

Because K, I, H, and A are upon AK it can readily be shown that there is a point P upon AK such that $7p=2k+2i+2h+a=a+b+c+d+e+f+g$, and that this point is independent of the direction of RQ. In a similar manner it may be shown that there is a similar point p' upon the diameter through G, such that $7p'=a+b+c+d+e+f+g$. Hence $p=p'$. But this cannot be true without regard to the direction of RQ unless P and P' are at the intersection of the two diameters, that is, at the center. Hence the perpendicular from O upon any line RQ is one-seventh the sum of the perpendiculars a, b, c, d, e, f, and g. Hence if RQ passes through O, $a+b+c+d+e+f+g=0$.

It is evident that a similar demonstration may be made for any regular polygon.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md., and CHARLES C. CROSS, Whaleyville, Va.

In a polygon of an even number of sides a diameter drawn through one of the vertices will pass through another vertex which will be symmetrical with reference to the center of the circle as the center of symmetry. Consequently, the sum of the perpendiculars drawn from the vertices on one side of any diameter will be equal to the sum of the perpendiculars drawn from the vertices of the

other side of that diameter. If the number of sides of the polygon is uneven there will be no such symmetry.

Let MN be a diameter, $ABCDEFG$ be a regular polygon of n sides, n being an odd number, and O the center. Draw all the radii to the vertices, then $\angle ACB=\angle BOC=\angle COD=\ldots\ldots=2\pi/n=\beta$. Denote $\angle AOM$ by a, $\angle GOM$ by a'. It will be perceived at once that on one side of the diameter MN there are $\frac{1}{2}(n-3)$ triangles, and on the other $\frac{1}{2}(n-1)$ triangles, not cut by the diameter.

Employing the well-known formula of the difference of two cosines, we have

$$\cos(a-\tfrac{1}{2}\beta)-\cos(a-\tfrac{1}{2}\beta)=2\sin a\sin\beta,$$

$$\cos(a+\tfrac{1}{2}\beta)-\cos(a+\tfrac{3}{2}\beta)=2\sin(a+\beta)\sin\tfrac{1}{2}\beta,$$

$$\cos(a+\tfrac{3}{2}\beta)-\cos(a+\tfrac{5}{2}\beta)=2\sin(a+2\beta)\sin\tfrac{1}{2}\beta$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$\cos[a+\tfrac{1}{2}(2m-1)\beta]-\cos[a+\tfrac{1}{2}(2m+1)\beta]=2\sin(a+m\beta)\sin\tfrac{1}{2}\beta.$$

Adding,

$$\frac{\sin(a+\frac{1}{2}m\beta)\sin\frac{1}{2}(m+1)\beta}{\sin\frac{1}{4}\beta}=\sin a+\sin(a+\beta)+\sin(a+2\beta)+\ldots\ldots+\sin(a+m\beta).$$

Putting the radius of the circle=1, this last formula gives us the sum of the perpendiculars drawn from the vertices upon MN on either side of the latter.

Putting $m=\frac{1}{2}(n-3)\beta$, this sum

$$=\frac{\cos(a_1-\frac{3}{4}\beta)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta}.$$

Putting $m=\frac{1}{2}(n-1)$ we get

$$\frac{\cos(a_1-\frac{1}{4}\beta)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta},$$

but $a_1=\beta-a$, therefore the latter expression is

$$\frac{\cos(\frac{3}{4}\beta-a)\cos\frac{1}{4}\beta}{\sin\frac{1}{2}\beta}.$$

Consequently we have equality for the two sums of the perpendiculars, which proves the theorem.

Also solved by *G. B. M. ZERR*.

123. Proposed by **ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.**

A étant le point d' intersection des médianes d' un triangle ABC, démontrerque $AB^2+BC^2+CA^2=3(GA^2+GB^2+GB^2)$. [Ex. 84, Géométrie. No. 2, 1re Anne *L' Éducation Mathématique.*]

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; CHARLES C. CROSS, Whaleyville, Va.; and H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Take $BC=a$, $AC=b$, and $AB=c$, and put the medians equal, respectively, m_a, m_b, and m_c. Then

$$m_a=\tfrac{1}{2}\sqrt{[2(b^2+c^2)-a^2]},$$
$$m_b=\tfrac{1}{2}\sqrt{[2(a^2+c^2)-b^2]},$$
$$m_c=\tfrac{1}{2}\sqrt{[2(a^2+b^2)-c^2]}.$$

But the medians of a triangle intersect at a common point two-thirds of the distance from the vertex to the middle of the opposite side. Whence

$$GA^2=(\tfrac{2}{3}m_a)^2=\tfrac{2}{9}b^2+\tfrac{2}{9}c^2-\tfrac{1}{9}a^2,$$
$$GB^2=(\tfrac{2}{3}m_b)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}c^2-\tfrac{1}{9}b^2,$$
$$GC^2=(\tfrac{2}{3}m_c)^2=\tfrac{2}{9}a^2+\tfrac{2}{9}b^2-\tfrac{1}{9}c^2.$$

$\therefore$ $GA^2+GB^2+GC^2=\frac{1}{3}(a^2+b^2+c^2)$, and $3(GA^2+GB^2+GC^2)=a^2+b^2+c^2=BC^2+AC^2+AB^2$.

II. Solution by J. W. YOUNG, Fellow and Assistant, Ohio State University, Columbus, O.; P. S. BERG, B. Sc., Principal of Schools, Larimore, N. D.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let ABC be the triangle, AM, BN, and CL the medians, and G the intersection of the medians. Then by a well-known theorem,

$$AC^2+CB^2=2CL^2+2.AL^2=2CL^2+\tfrac{1}{2}AB^2.$$

Similarly,

$$CB^2+BA^2=2BN^2+\tfrac{1}{2}AB^2,$$
$$BA^2+AC^2=2AM^2+\tfrac{1}{2}CB^2.$$

Adding and dividing by 2, we have

$$AC^2+CB^2+BA^2=CL^2+BN^2+AM^2+\tfrac{1}{4}(AB^2+AC^2+CB^2),$$
$$\text{or } AC^2+CB^2+BA^2+\tfrac{4}{3}(CL^2+BN^2+AM^2),$$

(since G divides medians in the ratio 2:1)

$$=\tfrac{4}{3}(\tfrac{9}{4}CG^2+\tfrac{9}{4}BG^2+\tfrac{9}{4}AG^2)=3(CG^2+BG^2+AG^2.$$

The same propositions can very easily be proven analytically.

Solved in a similar manner by *COOPER D. SCHMITT*, *G. B. M. ZERR*, *WALTER H. DRANE*, and *CHAS. C. CROSS*.

124. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Every conic that passes through all the foci of a conic is a rectangular hyperbola. [From Charlotte A. Scott's *Modern Analytical Geometry*.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If $u=0$......(1) be the general equation of the second degree, we have the well-known equations

$$Fx+Gy-H=Cxy\ldots\ldots(2),$$

$$2Gx-2Fy-A+B=C(x^2-y^2)\ldots\ldots(3),$$

for the foci, A, B, etc., being, as usual, the various minors of

$$\begin{vmatrix} a, & h, & g \\ h, & b, & f \\ g, & f, & c \end{vmatrix}$$

Either (2) or (3) is the rectangular hyperbola.

II. Solution by G. B. M. ZERR, A. M., Ph. D.. Professor of Mathematics and Science, Chester High School, Chester, Pa.

"If through each of the two imaginary points at infinity on any circle two tangents be drawn to the conic, these tangents will form a quadrilateral whose vertices will be the real and imaginary foci of the conic."

Both Puckle and Salmon solve this problem. Salmon's proof expanded is as follows:

Find the condition that $x-x'+(y-y')\sqrt{(-1)}=0$ should touch the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The condition that $mx+ny+p=0$ touches the same conic is

$$(bc-f^2)m^2+(ca-g^2)n^2+(ab-h^2)p^2+2(gh-af)pn+2(hf-bg)pm$$
$$+2(fg-ch)mn=0,$$
$$\text{or } Am^2+Bn^2+Cp^2+2Fpn+2Gpm+2Hmn=0.$$

Now $m=1$, $n=\sqrt{-1}$, $p=-x'-y'\sqrt{-1}$ or dropping the accents $p=-(x+y\sqrt{-1})$.

$\therefore A-B+Cx^2-Cy^2+2Cxy\sqrt{(-1)}-2Fx\sqrt{(-1)}+2Fy-2Gx-2Gy\sqrt{(-1)}+2H\sqrt{(-1)}=0.$

$\therefore C(x^2-y^2)+2Fy-2Gx+A-B=0\ldots\ldots(1),$

and $2Cxy-2Fx-2Gy+2H=0$, or

$$Cxy-Fx-Gy+H=0\ \ldots\ldots(2).$$

The intersections of (1) and (2), both of which are rectangular hyperbolas, determine the foci.

NOTE ON PROBLEM 118.

BY W. H. CARTER, VICE PRESIDENT CENTENARY COLLEGE, JACKSON, LA.

The solution of this problem published on page 241 of the October (1899) number is somewhat unsatisfactory as it involves the Differential Calculus.

I use the figure given. Pass a circle through the points D and C, and tangent to the line FH. Let E be the point of tangency. Then E is the required point. For at any other point of FH, as M, the $\angle CMD$ has a numerical measure equivalent to one-half of the difference between those of the arc CD and the other arc intercepted by its sides, while at E, the $\angle CED$ has the same measure as one-half the arc CD. The visual angle is therefore a maximum at E. The calculation of the distance FE is then easy.

NOTE.—Prof. J. W. Scroggs, of Eureka, Kas., sent in a neat and brief construction of problem 118, but it came too late for credit in the proper issue.

CALCULUS.

93. Proposed by JOHN R. JEFFREY, Student in Ohio State University, Columbus, Ohio.

Solve the following differential equation :

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+y=2x, \text{ when } x<1.$$

I. Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

Let $x=\sin\theta$, then $d\theta/dx=\sec\theta$.

$$\frac{d\theta}{dx}=\frac{dy}{d\theta}\cdot\frac{d\theta}{dx}=\sec\theta\frac{dy}{d\theta}.$$

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\sec\theta\frac{dy}{d\theta}\right)=\frac{d}{d\theta}\left(\sec\theta\frac{dy}{d\theta}\right)\frac{d\theta}{dx}=\sec^2\theta\tan\theta\frac{dy}{d\theta}+\sec^2\theta\frac{d^2y}{d\theta^2}.$$

Substituting, our equation becomes

$$\frac{d^2y}{d\theta^2}+y=2\sin\theta\ldots\ldots\ (1).$$

Consider $\dfrac{d^2y}{d\theta^2}+y=0$.

$y=\cos\theta$ is a particular solution. Now let $y=z\cos\theta\ldots\ldots(2)$.
Differentiating we get

$$\frac{d^2y}{d\theta^2}=\cos\theta\frac{d^2z}{d\theta^2}-2\sin\theta\frac{dz}{d\theta}-z\cos\theta.$$

Substitute this, and y from (2), in (1), and we get

$$\frac{d^2z}{d\theta^2}-2\tan\theta\frac{dz}{d\theta}=z\tan\theta\ldots\ldots\ (3).$$

Let $p=dz/d\theta$. Then (3) becomes

$$\frac{dp}{d\theta}-2\tan\theta.p=z\tan\theta\ \ldots\ldots(4).$$

Consider now $\dfrac{dp}{d\theta}-2\tan\theta.p=0$. Its solution is $p\cos^2\theta=c\ldots\ldots(5)$.

Differentiate this, regarding c as variable, and we get

$$\frac{dp}{d\theta}=2\tan\theta.p+\frac{1}{\cos^2\theta}\frac{dc}{d\theta}.$$

This in (4) gives $\dfrac{dc}{d\theta}=\sin2\theta\ldots\ldots(6)$.

$\therefore\ c=-\frac{1}{2}\cos2\theta+k_1$. Substitute this value of c in (5) and we get

$$\frac{dz}{d\theta}=-1+\tfrac{1}{2}(1+2k_1)\sec^2\theta.$$

$\therefore\ z=-\theta+\frac{1}{2}(1+2k)\tan\theta+k_2$. This value of z in (2) gives

$$y=-\theta\cos\theta+\tfrac{1}{2}(1+2k)\sin\theta+k_2\cos\theta.$$

Substitute for θ and we get

$$y=-\sin^{-1}x.\sqrt{(1-x^2)}+\tfrac{1}{2}x(1+2k)+k_2\sqrt{(1-x^2)},$$

or in a more general form

$$y=Ax+B\sqrt{(1-x^2)}-\sqrt{(1-x^2)}\sin^{-1}x,$$

which is the complete primitive.

II. **Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.**

Let $y=vx$, then the equation becomes

$$x(1-x^2)\frac{d^2v}{dx^2}+(2-3x^2)\frac{dv}{dx}=2x.$$

$$\therefore\ x^2(1-x^2)^{\frac{1}{2}}\frac{dv}{dx}=\sin^{-1}x-x\sqrt{(1-x^2)}+A.$$

$$\therefore\ \frac{dv}{dx}=\frac{\sin^{-1}x}{x^2(1-x^2)^{\frac{1}{2}}}+\frac{A}{x^2(1-x^2)^{\frac{1}{2}}}-1/x.$$

$\therefore\ vx=Bx-(A+\sin^{-1}x)\sqrt{(1-x^2)}$.

$\therefore\ y=Bx-(A+\sin^{-1}x)\sqrt{(1-x^2)}$.

III. Solution by M. C. STEVENS, M. A., Professor of Mathematics, Purdue University, Lafayette, Ind., and GEORGE LILLEY, Ph.D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

This equation is exact and the first integral is

$$(1-x^2)\frac{dy}{dx}+xy=x^2+c_1,\text{ or }\frac{dy}{dx}+\frac{x}{1-x^2}y=\frac{x^2}{1-x^2}+\frac{c_1}{1-x^2}.$$

This is a linear equation in which

$$\int Pdx=\int\frac{xdx}{1-x^2}=\log\frac{1}{\sqrt{(1-x^2)}}.$$

$$\therefore\frac{1}{\sqrt{(1-x^2)}}y=\int\frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}+c_1\int\frac{dx}{(1-x^2)^{\frac{3}{2}}}.\quad\text{Whence } x<1.$$

$$\frac{1}{\sqrt{(1-x)^2}}y=\frac{x}{\sqrt{(1-x^2)}}-\sin^{-1}x+\frac{c_1x}{\sqrt{(1-x^2)}}+c_2,$$

or $y=cx+\sqrt{(1-x^2)}(c_2-\sin^{-1}x)$ when c is written for c_1+1.

This example is found on page 144, *Johnson's Differential Equations.*

IV. Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

First solve $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}+y=0$.

Putting $\dfrac{dy}{dx}=p$, $(1-x^2)\dfrac{dp}{dx}-px+y=0$.

Differentiating, $(1-x^2)\dfrac{d^2p}{dx^2}-3x\dfrac{dp}{dx}=0$.

Putting q for $\dfrac{dp}{dx}$, $(1-x^2)\dfrac{dq}{dx}-3xq=0$.

Then $q=A(1-x^2)^{\frac{3}{2}}$, $p=\dfrac{Ax}{\sqrt{(1-x^2)}}+B$, $y=-A(1-x^2)^{\frac{1}{2}}+Bx$.

The particular integral may be found by the method of variation of parameters. Considering A and B variable, and substituting in the given equation,

$$-(1-x^2)^{\frac{3}{2}}\frac{d^2A}{dx^2}+(1-x^2)x\frac{d^2B}{dx^2}+x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}-x^2\frac{dB}{dx}=2x.$$

We may assume

$$x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}-x^2\frac{dB}{dx}=0.$$

Differentiating this and eliminating the second derivatives,

$$x(1-x^2)^{\frac{1}{2}}\frac{dA}{dx}+(1-x^2)\frac{dB}{dx}=-2x.$$

Solving for $\frac{dA}{dx}$ and $\frac{dB}{dx}$,

$$\frac{dB}{dx}=-2x,\ \frac{dA}{dx}=\frac{-2x^2}{(1-x^2)^{\frac{1}{2}}}.$$

Then $B=-x^2$, $A=\sin^{-1}x+x\sqrt{(1-x^2)}$.
The complete solution is then

$$y=(\sin^{-1}x-a)\sqrt{(1-x^2)}+(b-1)x,$$

a and b being arbitrary.

94. **Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.**

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis. [Ex. 16, page 166, *Rice and Johnson's Differential Calculus.*]

Solution by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

The equation of the tangent to an ellipse in terms of the slope being

$$y=mx+\sqrt{(a^2m^2+b^2)},$$

the altitude of the triangle will be the intercept of the tangent on the y-axis plus the semi-minor axis or $b+\sqrt{(a^2m^2+b^2)}$.

The base of the triangle found by putting $y=-b$ in the equation of the tangent is

$$\frac{-b-\sqrt{(a^2m^2+b^2)}}{m}.$$

The function of which a minimum value is to be found is accordingly

$$\frac{[b+\sqrt{(a^2m^2+b^2)}]^2}{m}$$

The derivative of this function equated to zero gives after reduction

$$a^4m^4-3a^2b^2m^2=0.$$

The value of m is $\pm b/a\sqrt{3}$, and $a^2m^2+b^2=4b^2$.

The intercept on the y-axis is therefore $2b$, and the altitude of the triangle is $3b$.

MECHANICS.

NOTE.—Prize Problem No. 86, is open for solution as only one solution has been received, and that one is faulty.

93. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A small rope, which is passed over a smooth pully, has attached at one end a weight of twenty pounds, and at the other end hangs a monkey, also weighing twenty pounds. Is it possible for the monkey to climb to the pulley, and if so, what will happen to the weight?

Solution by the PROPOSER.

When a body is in motion, its weight is altered, becoming greater when the body moves upward, and less when the body moves downward, and this increase or decrease in weight is proportional to the velocity. By this principle, when the monkey starts upward with a velocity, $ds/dt=\frac{1}{20}.Ft$, where F is the force with which he raises himself, his weight is increased by an amount proportional to $\frac{1}{20}Ft$. This force reacts undiminished upon the weight (as the pulley is smooth) and as the mass of the weight is also 20 pounds it must also begin to ascend with the same velocity as the monkey. This in turn increases the weight of the weight until it equals that of the monkey, which counteracts the effect of the monkey's increase in weight, and hence we have the result that monkey and weight will ascend with equal velocities.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

In a parallelogram $ABCD$, $\angle D=\beta$, $AB=a$, $BC=b$, the principal moments of inertia at the centroid are $(\frac{1}{24}m)[a^2+b^2\pm\sqrt{(a^4-b^4+2a^2b^2\cos2\beta)}]$ and the principal axes at the same point make with the side CD an angle θ given by

$$\tan2\theta=\frac{b^2\sin2\beta}{a^2+b^2\cos2\beta}.$$

Solution by the PROPOSER.

Let $ABCD$ be the parallelogram, O the intersection of the diagonals AC, BD, and OX, OY be the axes. For integration we will transform to OX, OY'.

$$\tan2\theta=\frac{2\Sigma mxy}{\Sigma mx^2-\Sigma my^2}.$$

$\Sigma mx^2=$moment of inertia about OX

$$=\rho\sin\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}(x+y\cos\beta)^2dxdy=\tfrac{1}{12}\rho ab\sin\beta(a^2+b^2\cos^2\beta)=\tfrac{1}{12}m(a^2+b^2\cos^2\beta).$$

$$\Sigma my^2=\rho\sin^3\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}y^2dxdy=\tfrac{1}{12}mb^2\sin^2\beta.$$

$$\Sigma mxy = \rho\sin^2\beta\int_{-\frac{1}{2}a}^{\frac{1}{2}a}\int_{-\frac{1}{2}b}^{\frac{1}{2}b}(x+y\cos\beta)y\,dx\,dy = \tfrac{1}{24}mb^2\sin2\beta.$$

$$\therefore\ \tan2\theta = \frac{b^2\sin2\beta}{a^2+b^2(\cos^2\beta-\sin^2\beta)} = \frac{b^2\sin2\beta}{a^2+b^2\cos2\beta}.$$

Let A, B be the principal moments.

$\therefore\ A\cos^2\theta + B\sin^2\theta = \frac{1}{12}m(a^2+b^2\cos^2\beta)\ldots\ldots(1)$.

$A\sin^2\theta + B\cos^2\theta = \frac{1}{12}mb^2\sin^2\beta\ldots\ldots(2)$.

$(1)+(2)$ gives $A+B=\frac{1}{12}m(a^2+b^2)$.

$(1)-(2)$ gives $A-B=\frac{1}{12}m(a^2+b^2\cos2\beta)\sec2\theta$

$$=\tfrac{1}{12}m\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}.$$

$\therefore\ A=\frac{1}{24}m[a^2+b^2+\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}]$.

$B=\frac{1}{24}m[a^2+b^2-\sqrt{(a^4+b^4+2a^2b^2\cos2\beta)}]$.

DIOPHANTINE ANALYSIS.

76. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

It is required to find four positive numbers such that if each be diminished by twice the cube of their sum the four remainders will be rational cubes.

Solution by the PROPOSER.

Let u, v, x, y be the numbers.

Then $u-2(u+v+x+y)^3=a^3/h^3(x+y+u+v)^3$, suppose.

$v-2(u+v+x+y)^3=b^3/h^3(x+y+u+v)^3$, suppose.

$x-2(u+v+x+y)^3=c^3/h^3(x+y+u+v)^3$, suppose.

$y-2(u+v+x+y)^3=d^3/h^3(x+y+u+v)^3$, suppose.

Adding we get

$$u+v+x+y-8(u+v+x+y)^3=\frac{a^3+b^3+c^3+d^3}{h^3}(u+v+x+y)^3.$$

Let $a^3+b^3+c^3+d^3=h^3$.

$\therefore\ u+v+x+y=9(u+v+x+y)^3$. $\quad\therefore\ u+v+x+y=\frac{1}{3}$.

$$\therefore\ u=\frac{a^3+2h^3}{27h^3},\quad v=\frac{b^3+2h^3}{27h^3},\quad x=\frac{c^3+2h^3}{27h^3},\quad y=\frac{d^3+2h^3}{27h^3}.$$

Let $a=1$, $b=5$, $c=7$, $d=12$, $h=13$.

$\therefore\ u=\frac{4395}{59319},\ v=\frac{4519}{59319},\ x=\frac{4737}{59319},\ y=\frac{6122}{59319}$.

Let $a=4,\ b=7,\ c=8,\ d=17,\ h=18$.

$\therefore\ u=\frac{11728}{157464},\ v=\frac{12007}{157464},\ x=\frac{12176}{157464},\ y=\frac{16577}{157464}$.

Other values can be found for $u,\ v,\ x,\ y$.

77. **Proposed by JOHN M. COLAW, A. M., Monterey, Va.**

Find (1) three consecutive numbers whose sum is a cube, and (2) three consecutive numbers the sum of whose cubes is a cube.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

(1). Let $n-1$, n, and $n+1$ be any three consecutive numbers.

Then $(n-1)+n+(n+1)=3n=$a cube$=27m^3$.

Whence $n=9m^3$.

$\therefore\ 9m^3-1$, $9m^3$, and $9m^3+1$ are the general expressions for three consecutive numbers whose sum is a cube.

Take $m=1$; then $8+9+10=27=3^3$.

Take $m=2$; then $71+72+73=216=6^3$; etc.

(2). $(n-1)^3+n^3+(n+1)^3=3n^3+6n=$a cube$=27m^3$.

Whence $n^3+2n=9m^3$.

Put $m=an$; then $n^3+2n=9a^3n^3$.

Whence $n^2+2=9a^3n^2$; and $n^2=2/(9a^3-1)$.

To obtain n integral, a must be fractional.

Put $a=1/b$; then $n^2=2b^3/(9-b^3)$.

To avoid imaginary results, $b<2\frac{1}{12}$.

The only integral values that can be assigned to b are 1 and 2.

Take $b=1$; then $n=\frac{1}{2}$.

Whence $(-\frac{1}{2})^3+(\frac{1}{2})^3+(\frac{3}{2})^3=(\frac{3}{2})^3$.

Take $b=2$; then $n=4$.

Whence $3^3+4^3+5^3=6^3$.

This is the only set of three consecutive integers the sum of whose cubes is a cube.

Fractional values of b give fractional values for n.

When $b=0$, $n=0$.

Whence $(-1)^3+0^3+1^3=0^3$.

Also solved by *CHARLES C. CROSS*, *JOSIAH H. DRUMMOND*, *ALOIS F. KOVARIK*, *NELSON L. RORAY*, *J. SCHEFFER*, *ELMER SCHUYLER*, and *G. B. M. ZERR*.

AVERAGE AND PROBABILITY.

81. **Proposed by LON C. WALKER, Assistant in Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.**

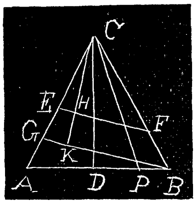
Find (1) the mean distance of all points on a side of an equilateral triangle from the opposite vertex; and (2), the average length of a line drawn at random across an equilateral triangle.

I. Solution by the PROPOSER.

1. Let each side$=a$ of the equilateral triangle ABC delineated in the annexed figure. Then its altitude $CD=\frac{1}{2}a\sqrt{3}$.

Put $DP=x$, then $CP=\sqrt{(x^2+\frac{3}{4}a^2)}$.

Hence the required mean distance is



$$M=\frac{\int_0^{\frac{1}{2}a}\sqrt{(x^2+\frac{3}{4}a^2)}dx}{\int_0^{\frac{1}{2}a}dx}=\tfrac{1}{4}a(2+3\log\sqrt{3}).$$

2. Suppose EF the random line. Draw GB parallel to EF, and CHK perpendicular to BG. Put $CH=x$ and $\angle CEF=\theta$. Then

$$EF=\frac{x\sin\frac{1}{3}\pi}{\sin\theta\sin(\frac{1}{3}\pi+\theta)}$$

$CK=a\sin\theta$, and the required average length is

$$L=\frac{\int_0^{\frac{1}{3}\pi}\int_0^{a\sin\theta}x\sin\frac{1}{3}\pi\operatorname{cosec}\theta\operatorname{cosec}(\frac{1}{3}\pi+\theta)d\theta dx}{\int_0^{\frac{1}{3}\pi}\int_0^{a\sin\theta}d\theta dx}$$

$$=a\int_0^{\frac{1}{3}\pi}\sin\tfrac{1}{3}\pi\sin\theta\operatorname{cosec}(\tfrac{1}{3}\pi+\theta)d\theta=\tfrac{1}{4}a\int_0^{\frac{1}{3}\pi}[\sqrt{3}-3\cot(\tfrac{1}{3}\pi+\theta)]d\theta=\tfrac{1}{12}\pi a\sqrt{3}.$$

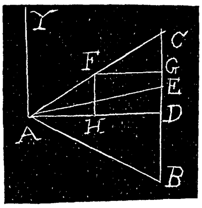
II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(1). Let $AC=2a$, $DE=x$. Then $AD=a\sqrt{3}$, $AE=\sqrt{(3a^2+x^2)}$.

$$\therefore M=\frac{\int_0^a\sqrt{(3a^2+x^2)}dx}{\int_0^a dx}=\frac{1}{a}\int_0^a\sqrt{(3a^2+x^2}dx.$$

$\therefore M=\frac{1}{4}a(4+3\log3)$.

If we regard $\angle DAE=\theta$ as the variable,



$$M=a\sqrt{3}\frac{\int_0^{\frac{1}{6}\pi}\sec\theta d\theta}{\int_0^{\frac{1}{6}\pi}d\theta}=\frac{6a\sqrt{3}}{\pi}\int_0^{\frac{1}{6}\pi}\sec\theta d\theta.$$

$$\therefore M=\frac{3a\sqrt{3}\log3}{\pi}.$$

(2). Let FG be the random line, $AH=x$, $DG=y$.
Then $FH=\frac{1}{3}\sqrt{3}x$, $FG=\sqrt{[(a\sqrt{3}-x)^2+(y-\frac{1}{3}\sqrt{3}x)^2}=l$.

$$\triangle=\text{average length}=\frac{\int_0^{a\sqrt{3}}\int_0^a ldxdy}{\int_0^{a\sqrt{3}}\int_0^a dxdy}$$

$$\triangle=\frac{1}{a^2\sqrt{3}}\int_0^{a\sqrt{3}}\int_0^a\sqrt{[(a\sqrt{3}-x)^2+(y-\tfrac{1}{3}\sqrt{3}x)^2]}dxdy$$

$$=\frac{1}{12a^2\sqrt{3}}\int_0^{a\sqrt{3}}\Big[4(a\sqrt{3}-x)^2+x\sqrt{[9a^2+(4x-3\sqrt{3}\,a)^2]}$$

$$-6(a\sqrt{3}-x)^2\log\left(\frac{\sqrt{[9a^2+(4x-3\sqrt{3}\,a)^2]}-2x}{6(\sqrt{3}-x)}\right)\Big]dx.$$

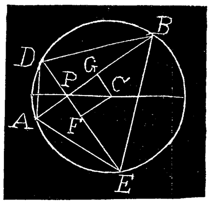
$\therefore\ \triangle=(a/16)[6+2\sqrt{3}+4\log3+\log(3+2\sqrt{3})]$.

82. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Find the average area of the quadrilateral formed by joining the extremities of two chords perpendicular to each other and passing through a point at a distance a from the center of a circle radius R.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and the PROPOSER.

Let $\triangle$=average area, $\angle BPC=\theta$, P the given point, C the center of the cirle, $CP=a$, $CB=R$.



CASE I. $a<R$, $CG=a\sin\theta$, $CF=a\cos\theta$.

$\therefore\ AB=2\sqrt{(R^2-a^2\sin^2\theta)}$, $DE=2\sqrt{(R^2-a^2\cos^2\theta)}$.

Area $ADBE=2\sqrt{[(R^2-a^2\sin^2\theta)(R^2-a^2\cos^2\theta)]}$
$=2R^2\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}$ where $e^2=a^2/R^2$.

$$\therefore\ \triangle=2R^2\int_0^{\frac{1}{2}\pi}\sqrt{[(1-e^2\sin^2\theta)(1-e^2\cos^2\theta)]}d\theta\Big/\int_0^{\frac{1}{2}\pi}d\theta$$

$$=\frac{2R^2}{\pi}(2-e^2)\,E\left(\frac{e^2}{2-e^2},\ \tfrac{1}{2}\pi\right).$$

(See page 356, No. 10, Vol. I, for above integration.

CASE II. $a>R$ and $a<R\sqrt{2}$, $PF=a\cos\theta$, $PG=a\sin\theta$.

$PB=a\cos\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PA=a\cos\theta-\sqrt{(R^2-a^2\sin^2\theta)}$.

$PE=a\sin\theta+\sqrt{(R^2-a^2\sin^2\theta)}$, $PD=a\sin\theta-\sqrt{(R^2-a^2\cos^2\theta)}$.

Area $ABED=\frac{1}{2}(BP.PE-AP.PD)$

$$=a[\sqrt{(R^2-a^2\sin^2\theta)}\sin\theta+\sqrt{(R^2-a^2\cos^2\theta)}\cos\theta].$$

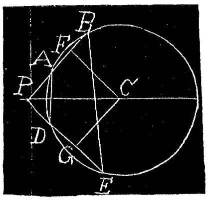
The limits of θ are $\frac{1}{2}\pi-\sin^{-1}(R/a)=\theta''$ and $\sin^{-1}(R/a)=\theta'$.

$$\therefore \ \triangle = aR\int_{\theta''}^{\theta'}[\sqrt{(1-e^2\sin^2\theta)}\sin\theta+\sqrt{(1-e^2\cos^2\theta)}\cos\theta]d\theta\Big/\int_{\theta''}^{\theta'}d\theta$$

$$=\frac{aR}{4\sin^{-1}(R/a)-\pi}\Big[\sin\theta'\sqrt{(1-e^2\cos^2\theta')}$$

$$-\sin\theta''\sqrt{(1-e^2\cos^2\theta'')}-\cos\theta'\sqrt{(1-e^2\sin^2\theta')}$$

$$+\cos\theta''\sqrt{(1-e^2\sin^2\theta'')}$$



$$+\frac{1-e^2}{e}\log\left(\frac{e\sin\theta'+\sqrt{(1-e^2\cos^2\theta')}}{e\sin\theta''+\sqrt{(1-e^2\cos^2\theta'')}}\right)$$

$$-\frac{1-e^2}{e}\log\left(\frac{e\cos\theta'+\sqrt{(1-e^2\sin^2\theta')}}{e\cos\theta''+\sqrt{(1-e^2\sin^2\theta'')}}\right)\Big].$$

But $\sin\theta'=\cos\theta''=R/a$, $\sin\theta''=\cos\theta'=[\sqrt{(a^2-R^2)}]/a$.
$e^2\sin^2\theta'=e^2\cos^2\theta''=1$, $e^2\sin^2\theta''=e^2\cos^2\theta'=[(a^2-R^2)/R^2]=e^2-1$.
Whence by substitution and reduction we get

$$\triangle=\frac{R^2}{2\sin^{-1}(R/a)-\frac{1}{2}\pi}\left[\sqrt{(2-e^2)}+(1-e^2)\log\left(\frac{1+\sqrt{(2-e^2)}}{\sqrt{(e^2-1)}}\right)\right]$$

83. Proposed by **F. P. MATZ, M. Sc., Ph. D.**, Professor of Mathematics and Astronomy, Irving College Mechanicsburg, Pa.

Find the average area of all ellipses whose semi-axis major is a.

I. Solution by **J. W. YOUNG**, Fellow and Assistant, Ohio State University, Columbus, Ohio.

The area of an ellipse whose major-axis is a, and whose minor-axis is b, is πab. We must find the average of all possible values of this expression as b varies from zero to a.

$$\therefore \text{ Average required}=\frac{\pi a\int_0^a bdb}{\int_0^a db}=\tfrac{1}{2}\pi a^2,$$

$=\frac{1}{2}$ the area of the circle whose radius is the major-axis of the ellipse.

II. Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.

1. Let x=semi-conjugate axis. Then average area

$$=\pi a\frac{\int_0^a xdx}{\int_0^a dx}=\tfrac{1}{2}\pi a^2=1.5708a^2.$$

2. Let e=eccentricity. Then area$=\pi a^2\sqrt{(1-e^2)}$.

$$\therefore \text{ Average area}=\pi a^2 \frac{\int_0^1 \sqrt{(1-e^2)}}{\int_0^1 de}=\tfrac{1}{4}\pi^2a^2=2.4674a^2.$$

MISCELLANEOUS.

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73. Proposed by CHAS. E. MYERS, Canton, Ohio.

In an ice cream freezer, cream of a homogeneous character and at the uniform temperature of 60° Fahrenheit is put into a cylinder having a closed base, and the whole put into a freezing mixture so as to subject the base and convex surface to a constant temperature of 30° Fahrenheit. Required the temperature at any point within the cream after the expiration of a given time. [From *Higher Mathematics*.]

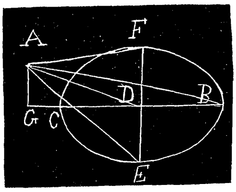
No solution of this problem has been received.

74. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at D. Find in an indefinite vertical plane passing through CB, a point $A=5$ feet$=c$ from D, the ellipse being seen from A as a circle.

I. Solution by the late B. F. BURLESON, and the PROPOSER.

The eye being at A, and the ellipse being projected as a circle, CB and EF subtend equal angles at A, or $\angle EAF=\angle BAC$. Produce DC to G, A being vertically over G, and put $CG=x$, and $GA=y$, and $\angle ADC=\phi$=angle of elevation of A.



Then $y=\sqrt{[c^2-(a+x)^2]}$ ......(1).

$AB=\sqrt{[(2a+x)^2+y^2]}$ .....(α).

$\sin\angle ACG=\sin\angle ACB=y/\sqrt{(x^2+y^2)}$......($\beta$), and $\tan\angle EAD=b/c$ ....(γ).

$\therefore \sin\angle EAF=\sin\angle BAC=2dc/(b^2+c^2)$..... ($\delta$).

From $\triangle BAC$ we have the proportion, $AB : \sin\angle ACB :: BC : \sin\angle BAC$.

$$\therefore \frac{2bc\sqrt{[(2a+x)^2+y^2]}}{b^2+c^2}=\frac{2ay}{\sqrt{(x^2+y^2)}}......(2).$$

Resolving (1) and (2) we have $x=\dfrac{c\sqrt{[(c^4-a^2b^2)(a^2-b^2)]}-a}{a(c^2-b^2)}$

$$=\tfrac{5}{63}\sqrt{(2945)}-3=1.30697255 \text{ feet}.$$

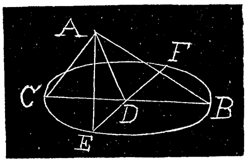
$$\therefore y=\frac{bc(c^2-a^2)}{a(c^2-b^2)}=2\tfrac{34}{63} \text{ feet}.$$

$$\phi=\sin^{-1}\left(\frac{b(c^2-a^2)}{a(c^2-b^2)}\right)=\sin^{-1}(\tfrac{32}{63})=30^\circ\ 31'\ 35\tfrac{1}{4}''.$$

II. **Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, Ohio.**

If the ellipse is seen from A as a circle, $\angle EAF=\angle CAB$. This relation enables us to calculate the coördinates of the point A, in the vertical plane through CB, the origin being at D, and CB being the axis of x.

$$\tan\tfrac{1}{2}\angle EAF=b/c.$$



Also let $AC=q$ and $AB=p$; $CB=2a$.

Then $\tan\frac{1}{2}\angle CAB=\sqrt{\left(\frac{(s-p)(s-q)}{s(s-2a)}\right)}$

where $s=\frac{1}{2}(p+q+2a)$.

Since $\angle EAF=\angle CAB$, we have, after reducing,

$$\frac{b^2}{c^2}=\frac{4a^2-(p-q)^2}{(p+q)^2-4a^2}\ldots\ldots(1).$$

Now, since $AD=c$ is the median of $\triangle ABC$, we have by a common trigonometrical formula,

$$2c^2=p^2+q^2-2a^2 \text{ or } p^2+q^2=2(a^2+c^2)\ldots\ldots(2).$$

From equations (1) and (2), we obtain

$$p+q=\pm2\sqrt{\left(\frac{a^2b^2-c^4}{b^2-c^2}\right)};\ p-q=\pm2c\sqrt{\left(\frac{b^2-a^2}{b^2-c^2}\right)}.$$

Now, to obtain x and y, we have the relations,

$$(a+x)^2+y^2=p^2,\ (a-x^2)+y^2=q^2,$$

from which $x=\dfrac{p^2-q^2}{4a}$.

Substituting from above, we have

$$x=\frac{\pm c}{a(b^2-c^2)}\sqrt{[(a^2b^2-c^4)(b^2-a^2)]},\quad y=\pm\frac{bc(a^2-c^2)}{a(b^2-c^2)}.$$

In the special case given, where $a=3$, $b=2$, $c=5$, we have

$$x=\pm4.31 \text{ feet},\quad y=\pm2.54 \text{ feet}.$$

NOTE.—Other solutions of this problem will appear next month.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

126. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Bought 150 head of stock for \$300, paying for each kind $\$2\frac{5}{6}$, $\$1\frac{5}{9}$, and $\$\frac{5}{7}$, respectively. Find number of each kind bought.

127. Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.

A man borrows \$1000 of a Building and Loan Association, and at the same time sub-subscribes for 10 \$100-shares of stock. A membership fee of \$1 per share is charged. At the beginning of each month an installment of \$1 per share is paid, also 5% interest and 5% premium on the \$1000. The stock matures in 75 months and the debt is cancelled. What rate of interest does he pay per annum?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than April 10.

GEOMETRY.

138. Proposed by JOHN M. HOWIE, Professor of Mathematics, The Nebraska State Normal, Peru, Neb.

K is the middle point of any chord AB of a given circle. CD and EF are any two chords passing through K. CF and ED intersect AB at M and N, respectively. Prove that KM equals KN.

139. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If $x^2+y^2=1$ [x and y being points corresponding to complex numbers], prove that x and y are at the ends of conjugate radii of an ellipse whose foci are ± 1. [From *Harkness and Morley's Introduction to the Theory of Functions.*]

140. Proposed by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.

Having given two points on a range and a point that bisects the distance between two other points that form an harmonic ratio with the given points, give, if possible, a geometrical construction for locating the other two points.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than April 10.

MECHANICS.

103. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Given the lengths a, b of the sides of a parallelogram, the direction of side a, and the position of the centroid. Prove that the locus of the foci of the ellipse of gyration at the centroid is a Cassinian Oval, having its foci distant $a/2\sqrt{3}$ from the centroid, and the constant product of its focal distances equal to $\frac{1}{12}b^2$.

104. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

From a locomotive and tender standing still on a bridge, the pressure on the bridge is $p_1=80$ tons. The track is supposed to be straight and practically

horizontal. Had the locomotive and tender been running at the rate of $r=60$ miles an hour, how many tons would the pressure on the bridge have been ?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than April 10.

AVERAGE AND PROBABILITY.

91. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Six points A, B, C, D, E, F are taken at random on the surface of a sphere. Find the chance that the plane through A, B, C intersects the plane through $D, E. F$ within the sphere.

92. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.

A circular field, radius r, is divided into four *equal* parts, by concentric circles and three concentric rings. From the center of this field are fired *at random*, and with such a velocity as not to produce a range greater than the radius of the field, $m=1000$ projectiles of the *same* kind. How many projectiles should have fallen into each one of these four equal parts of the field ?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than April 10.

EDITORIALS.

Mr. W. D. Cairns has been appointed to an instructorship in mathematics in Oberlin College.

A fac-simile reprint of Legendre's Theorie des Numbers has lately been issued by A. Hermann, of Paris.

Among the lecturers before the students of the College for Women of the Western Reserve University, is Prof. E. S. Loomis, Teacher of Mathematics in the West High School of Cleveland, Ohio. Prof. Loomis's lectures are on Fundamentals in the Teaching of Arithmetic, and Essentials in Teaching Algebra.

BOOKS AND PERIODICALS.

Elements of Precise Surveying and Geodesy. By Mansfield Merriman, Professor of Civil Engineering in Lehigh University. 8vo. Cloth, 261 pages. Price, $2.50. New York : John Wiley & Sons.

The work begins with an elementary treatment of the method of least squares, developing the theory of the method in such an elegant and lucid way as to be clearly comprehended by a beginner. Many examples are solved to illustrate the various principles as they are developed. One not familiar with the Law of Probability of Error, and the Method of Least Squares and desiring to get a working knowledge of the subject needs this book. Chapter II treats of Precise Plane Triangulation in which is applied the Method of Least Squares for the correction of measured magnitudes ; Chapter III treats of Base

Lines; Chapter IV, of Leveling; Chapter V, of Astronomical Work, in which is treated the Precise Measurement of Latitude by various methods; Longitude, Azimuth, and Time; Chapter VI treats of Spherical Geodesy. This subject is introduced by a brief but very interesting history of Geodesy, beginning with the earliest times and bringing it down to the present time. Chapter VII deals with Spheriodal Geodesy; Chapter VIII, Geodetic Coördinates and Projections, including the various map-projections; Chapter IX, of Geodetic Triangle; and Chapter X, on the Figure of the Earth. This last chapter is full of interest, as the various assumed forms of the earth are discussed briefly. These various forms are the Spheriod, the Ellipsoid, the Ovaloid, and the Geoid. The book is one of the highest interest and importance not only to engineers but to mathematicians as well.

B. F. F.

Statistical Methods with Special Reference to Biological Variation. By C. B. Davenport, Instructor in Zoölogy in Harvard University. 16mo. Morocco, 135 pages. Price, $1.25. New York: John Wiley & Sons.

This work is intended especially for Botanists, Zoölogists, Anthropologists, Anatomists, Physiologists, and Psychologists who are interests in the quantitative study of species and of organic variation. It will also be of service to Economists, Sociologists, Meteorologists, and practical Statisticians. It treats in simple language, and for the most part without the use of mathematics beyond the elements of algebra of the statistical methods elaborated by Galton and Pearson. The mean, mode, probable error, index of variation, the coefficients of correllation and heredity are defined and the methods of getting them explained. The treatment of curves of the different classes, normal, skew (of three types), compound, and multimodal, is made clear. The work is a complete handbook, for it contains tables of reduction from English to metric units; squares, cubes, roots, and reciprocals of numbers from 1 to 1000; six-place logarithms of numbers and circular functions; table of gamma functions, etc. The table of using each table is fully explained. There have been added also some pages of cross-section paper, a metric scale, and a protractor. The book contains 31 figures, is of pocket size, and is bound in morocco. B. F. F.

Transactions of the American Mathematical Society, Edited by Drs E. H. Moore, E. W. Brown, and T. S. Fiske. Published quarterly by the Society with the coöperation of Harvard, Yale, Princeton, Columbia, Northwestern, Cornell, The University of Chicago, Haverford College, Bryn Mawr College, and the University of Califonia.

The first number of the first volume contains the following articles: Conics and Cubics connected with a Plane Cubic by certain Covariant Relations, by H. S. White; Formenthoretische Entwickelung der Herrn White's Abhandlung über curven dritter Ordung enthallenten Sätze, by P. Gordan; Sur la définition générale des functions analytiques, d'apès Cauchy, by E. Goursat; On a Class of Particular Solutions of the Problem of Four Bodies, by F. R. Moulton; Definition of the Abelian, the Two Hypoabelian, and the Related Linear Groups as Quotient-groups of the Groups of Isomorphisms of Certain Elementary Groups, by L. E. Dickson; Note on the Unilateral Surface of Moebius, by H. Maschke; On Regular Singular Points of Linear Differential Equations of the Second Order whose Coefficients are not necessarily Analytic, by M. Bocher; The Elliptic Sigma-Functions considered as a Special Case of the Hyperelliptic Sigma-Functions, by O. Bolza; On Groups which are the Direct Products of Two Subgroups, by G. A. Miller; On Certain Crinckly Curves, by E. H. Moore; A Definition of the General Abelian Linear Group, by L. E. Dickson. B. F. F.

Annals of Mathematics. Published under the Auspices of Harvard University. Issued Quarterly. Price, $2.00 per year in advance.

The January (1900) number contains the following: On Three-Dimensional Determinants, by E. R. Hedrick; On Tide Currents in Estuaries and Rivers, by Ernest W. Brown; Note on Netto's Theory of Substitution, by Dr. G. A. Miller; A Method of Solving Determinants, by G. Macloski; The Development of Functions, by S. A. Corey; Illustrations of the Elliptic Integrals of the First Kind by a certain Link-Work, Dr. Arnold Emch; and Problems in the Theory of Continuous Groups, by Chas. L. Bouton.

B. F. F.

Periodico di Mathematica for January and February, contains among other

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VOL. VII. MARCH, 1900. No. 3.

ON THE TRANSCENDENTAL FORM OF THE RESULTANT.

By DR. E. D. ROE, Jr., Elmira, N. Y.

(A paper presented December 28, 1899, at the meeting of the Chicago Section of the American Mathematical Society.)*

I. DERIVATION AND PROPERTIES.

§ 1. INTRODUCTION.

The writer was incited to the following investigation by portions of letters written by Professor Gordan to M. Hermite and himself. Under date of October 17, 1898, in a letter to the writer Professor Gordan says: "Ich habe die Untersuchung der Resultante weiter verfolgt und die Formel gefunden, sie durch Potenzsummen s der beiden Gleichungen auszudruecken. Ich habe sie nach Paris gesandt, wo sie demnächst in den Comptes Rendus erscheinen wird. R sei die Resultante der Gleichungen

$$f=x^m+a_1x^{m-1}+a_2x^{m-2}+\ldots\ldots$$

$$1+\phi=1+b_1x+b_2x^2+\ldots\ldots$$

Die $s_\rho(a)$ sind die Potenzsummen der Wurzeln α der Gleichung f und die $s_\rho(b)$

*The writer wishes here to acknowledge his obligations to Prof. H. S. White for presenting this paper and for subsequent criticisms for the revision. He wishes also to thank Prof. M. Bocher for friendly criticism.

dieselben Functionen der b. Die Producte der s bezeiche ich durch S: $S_\lambda = (s_{\lambda_1})^{\rho_1}(s_{\lambda_2})^{\rho_2}\ldots$ und fuehre die Zahl g_λ ein, $g_\lambda = (\lambda_1)^{\rho_1}(\lambda_2)^{\rho_2}\ldots\rho_1!\,\rho_2!\ldots$ Es ist dann:

$$R = \Sigma \frac{(-1)^{\rho_1+\rho_2+\cdots\cdots}}{g_\lambda} S_\lambda(a)\, S_\lambda(b).$$

Die Summe erstreckt sich ueber alle λ, ρ, fuer welche

$$\lambda_1\rho_1 + \lambda_2\rho_2 + \cdots\cdots \leqq mn$$

ist.

Aus diesser Formel kann man leicht die Entwickelung der symmetrischen Functionen

$$\Sigma\alpha_1^{\kappa_1}\alpha_2^{\kappa_2}\alpha_3^{\kappa_3}\cdots\cdots = \beta_{\kappa_1}\beta_{\kappa_2}\beta_{\kappa_3}\cdots\cdots$$

sowohl als ganze Function der s als auch als ganz Function der Coefficienten a ableiten. In der Formel

$$\beta_{\kappa_1}\beta_{\kappa_2}\cdots\cdots\beta_{\kappa_\nu} = \Sigma(-1)^{\nu+\rho_1+\rho_2+\cdots\cdots} C_{\lambda,\kappa} S_\lambda$$

sind die Coefficienten $C_{\lambda,\kappa}$ mittelst der Formel gegeben:

$$C_{\lambda,\kappa} = \Sigma \frac{1}{\tau_1!\,\tau_2!\ldots\ldots}(\varkappa_{11}\varkappa_{12}\ldots\varkappa_{1\nu_1} \mid \varkappa_{21}\varkappa_{22}\ldots\varkappa_{2\nu_2} \mid \varkappa_{31}\varkappa_{32}\ldots\varkappa_{3\nu_3} \mid \ldots).$$

In ihr bedeutet der Klammerausdruck das Product$(\nu_1-1)!(\nu_2-1)!(\nu_3-1)!\ldots$ und die τ sowohl die Anzahl gleicher Zahlen in Combinationen $\varkappa_{\mu_1}\varkappa_{\mu_2}\ldots$ als auch die Anzahl von Gleichen Combinationen. Die Summe erstreckt sich ueber alle Combinationen der Zahlen $\varkappa_1\varkappa_2\ldots\varkappa_\nu$ bei denen die $\varkappa_{\mu_1}+\varkappa_{\mu_2}-\varkappa_{\mu_3}\ldots = \lambda_\mu$ sind. In der Formel

$$\beta_{\kappa_1}\beta_{\kappa_2}\ldots\beta_{\kappa_\nu} = \Sigma(-1)^{\nu+\nu_1+\rho_1+\rho_2+\cdots\cdots}\varkappa_1\varkappa_2\ldots \mid \mu_1\mu_2\ldots\; a_{\mu_1}a_{\mu_2}\ldots a_{\mu_{\nu_1}}$$

lassen sich die Coefficienten $\varkappa_1\varkappa_2\ldots \mid \mu_1\mu_2\ldots$ mittelst der Formel berechnen

$$\varkappa_1\varkappa_2\ldots \mid \mu_1\mu_2\ldots = \Sigma g_\lambda \frac{1}{\tau_1!\,\tau_2!}(\varkappa_{11}\varkappa_{12}\ldots \mid \varkappa_{21}\varkappa_{22}\ldots \mid \ldots)$$

$$(\mu_{11}\mu_{12}\ldots \mid \mu_{21}\mu_{22}\ldots \mid \ldots)$$

wo die $\kappa_{\sigma_1}\kappa_{\sigma_2}\ldots\mu_{\sigma_1}\mu_{\sigma_2}\ldots$ Combinationen der $\varkappa$ und μ darstellen, welche durch die Relationen ver knuepft sind $\varkappa_{\sigma_1}+\varkappa_{\sigma_2}+\ldots = \mu_{\sigma_1}+\mu_{\sigma_2}+\ldots = \lambda_\sigma$.

Sie sehen in beiden Fällen geschieht die Berechnung durch independte Formeln und nicht durch Recursionen. Mit eineger Uebung genuegen bis zum Grade 8 diese Formeln zur Berechnung eines Coefficienten ca 3 Minuten. Recursionsformeln sind nicht mehr noethig."

It was not until November, 1899, that opportunity was found to look over the letter to M. Hermite (Comptes Rendus 127, 1898, page 539). On examination difficulties were found in it, which the writer in a letter, November 9, 1899, stated to Professor Gordan, together with a sketch of proposals for their removal. The privilege of publishing the letter of October 17, 1898, was also asked. In reply November 21, 1899, Professor Gordan wrote as follows: "Ihre Behandlung der Aufgabe ist richtig und stimmt im Wesentlichen mit derjenigen ueberein, welche ich in Muenchen vorgetragen habe. Meine Arbeit habe ich vor ca 6 Monaten an die Annalen gesandt, sie ist aber noch nicht gedruckt; sobald diess geschieht werde ich Ihnen ein Exemplar senden. Wenn sie selbst etwas darueber schreiben wollen, so wird mich das sehr freuen. Mine Darstellung in dem Briefe an Hermite war noch unreif; ich habe sie seither verbessert."

The writer has not yet received a copy of Professor Gordan's Munich paper, and what follows is substantially the elaboration of the sketch sent to Professor Gordan November 9, 1899. The working out of his results is given in much more detail than was needed in the sketch.

§ 2. THE FUNCTIONS f, ϕ, ψ.

The functions

$$f=x^m+a_1x^{m-1}+\ldots+a_m\ldots\ldots(1)$$

$$\phi=x^n+b_1x^{n-1}+\ldots+b_n\ldots\ldots(2)$$

$$\psi=1+b_1x+\ldots+b_mx^n\ldots\ldots(3)$$

are three integral functions of the mth and nth degrees, respectively. Let the roots of f be α_1, α_2, $\ldots\alpha_m$, and those of ϕ, β_1, β_2, $\ldots\beta_n$, then the roots of ψ are $1/\beta_1$, $1/\beta_2$, $\ldots 1/\beta_n$, *i. e.*, the roots of ψ are the reciprocals of roots of ϕ. We have:

$$\psi=(1-\beta_1x)(1-\beta_2x)\ldots(1-\beta_nx)\ldots\ldots(4).$$

§ 3. THE EXPONENTIAL FORM OF ψ.

By developing $\log\psi=\sum_{i=1}^{i=n}\log(1-\beta_ix)\ldots\ldots(5)$ which we may always certainly do, so long as we take x to satisfy the inequality $\mathrm{mod}(\beta x)<1$, where β denotes that root for which $\mathrm{mod}(\beta)\geqq\mathrm{mod}(\beta_i)$, we get

$$\log\psi=-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r\ldots\ldots(6)$$

where $s_r(b)$ denotes the sum of the rth powers of the β_i's, or we may write

$$\psi(x)=e^{-\sum_{r=1}^{r=\infty}(1/r)s_r(b)x^r}\ldots\ldots(7)$$

for values of x satisfying the inequality $\mathrm{mod}(x)<\mathrm{mod}(1/\beta)$, and this is the exponential form of ψ.

§ 4. THE RESULTANT OF f AND ψ.

If we choose those functions f, whose roots satisfy the inequality $\mathrm{mod}(\alpha)\leqq\mathrm{mod}(x)<\mathrm{mod}(1/\beta)$, where α is that root for which $\mathrm{mod}(\alpha)\geqq\mathrm{mod}(\alpha_i)$, then $\psi(\alpha_i)=e^{-\sum\limits_{r=1}^{r=\infty}(1/r)s_r(b)\alpha_i^r}$, and the resultant of f and ψ, $\pi_{i=1}^{i=m}\psi(\alpha_i)$, becomes

$$R=e^{-\sum\limits_{r=1}^{r=\infty}(1/r)s_r(a)s_r(b)}\ldots\ldots(8),$$

where $s_r(a)$ denotes the sum of the rth powers of the α_i's. The formula (8) is *the exponential form* of the resultant of f and ψ, and it has been shown that it is true for such functions f and ψ which are so related that the inequality $\mathrm{mod}(\alpha) < \mathrm{mod}\ (1/\beta)$ is satisfied.

§ 5. THE EXPANSION OF R.

If we expand the exponential form given in the last section we obtain :

$$R=\sum_{\nu=0}^{\nu=\infty}\frac{(-1)^\nu}{\nu\,!}\left(\sum_{\lambda=1}^{\lambda=\infty}(1/\lambda)s_\lambda\,(a)s_\lambda\,(b)\right)^\nu\ldots\ldots(9).$$

By the multinomial theorem,

$$\left(\sum_{\lambda=1}^{\lambda=\infty}(1/\lambda)s_\lambda\,(a)s_\lambda\,(b)\right)^\nu=\Sigma\frac{\nu\,!}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\ldots.\rho_1!\,\rho_2!\ldots.}s_{\lambda_1}{}^{\rho_1}(a)s_{\lambda_1}{}^{\rho_1}(b)s_{\lambda_2}{}^{\rho_2}(a)s_{\lambda_2}{}^{\rho_2}(b)\ldots.$$

where $\rho_1+\rho_2+\ldots.=\nu$, hence,

$$R=\sum_{\nu=0}^{\nu=\infty}\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots..}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\ldots.\rho_1!\,\rho_2!\ldots.}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\ldots\ldots(10).$$

We shall call this *the transcendental form* of the resultant. And we here add this remark : Though the previous developments upon which (10) depends have not been shown to be true for all functions f and ψ, yet when the left member R and the symmetric functions $s_r(a)$, $s_r(b)$ are replaced, for admissible functions, by their values in terms of the a's and b's, the relation (10) is seen to be an identity in terms of the latter, and therefore true for all functions f and ψ.

§ 6. COMPLETE GENERALITY OF THE TRANSCENDENTAL FORM.

From the foregoing sections it will be observed that the transcendental form of the resultant is precisely the same in terms of the s's for every binary resultant whatever. In obtaining the transcendental expression for R, we stated it is true with forms f and ψ of the mth and nth degrees, but we should have come precisely to the same transcendental form in terms of the s's, if we had used functions of any other degrees. This is the reason why we omitted to affect the symbol R with the subscripts m and n. *The transcendental form expresses*

all binary resultants formally in the same form, and is a universal formula comprehending them all. It is the most general formula for binary resultants.

§ 7. THE FINITE FORM OF THE RESULTANT.

As the resultant $R_{m,n}$ of two forms of the mth and nth degrees of the form f and ψ, must contain terms of weight mn in the a's, but no terms of higher weight, it follows that the terms of the transcendental form of weight greater than mn, taken together are, by the remark of § 5, identically equal to zero, when the s's are replaced by their values in terms of the a's and b's, though this will not appear explicitly or formally in terms of the s's,* and for the finite form we have,

$$R_{m,n}=\Sigma\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\cdots\cdots(11)$$

where the summation is to be extended over all values of the λ's and ρ's for which $\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot\leqq mn$.

Example : Let
$$f=x^2+a_1x+a_2$$
$$\phi=x+b_1$$
$$\psi=1+b_1x.$$

By the preceding formula or directly by elementary methods we find

$$R_{f,\,\psi}=R_{2,\,1}=1-s_1(a)s_1(b)+\tfrac{1}{2}(s_1(a)s_1(b))^2-\tfrac{1}{2}s_2(a)s_2)b).$$

The resultant of any two binary forms may be expressed in this manner by (11), for it is only necessary to supply for the $s(b)$'s those s's which belong to the form whose roots are the reciprocals of those of the given second form.

§ 8. GENERALITY OF THE FORMULA (11).

The form (11) which we may write, by using Gordan's abbreviations,

$$R_{m,\,n}=\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot}S_\lambda\,(a)\,S_\lambda\,(b)\quad\cdots\cdot(12)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot\leqq mn$, represents formally all resultants $R_{\mu,\,\nu}$, for which $\mu\nu=mn$, in terms of the s's in precisely the same form ; therefore it is still a very general and elastic form. In order to make it definite, one substitutes for the s's their values in terms of the a's and b's for definite equations whose degrees m_1, n_1, satisfy the condition $m_1n_1=mn$. Then it represents a definite resultant R_{m_1,n_1} of the given kind.

Example : $R_{12,1}$, $R_{1,12}$, $R_{3,\,4}$, $R_{4,\,3}$, $R_{2,\,6}$, and $R_{6,\,2}$, all have formally the same expression in terms of the s's.

The expression for $R_{m,n}$ will represent something which is not at all a re-

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*And as Professor White has suggested, it is also well to state, what is involved in this, that all terms of the same weight beyond those of weight mn, are seen to vanish identically by themselves, when the s's are replaced by a's and b's.

sultant (something more or less than a resultant) if values of the s's in terms of the a's and b's are substituted which belong to equations where $m_1 n_1$ not equal to mn. In consequence of this the symbol $R_{m,n}$ includes not only resultants but also other expressions not resultants. We use the symbol in the following section in the enlarged sense.

§ 9. PARTIAL DIFFERENTIAL EQUATION WITH RESPECT TO THE s'S.

If we differentiate the form

$$R_{m,\,n}=\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot}(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}(s_{\lambda_2}(a)s_{\lambda_2}(b))^{\rho_2}\cdots\cdots$$

of (11) partially with respect to $s_{\lambda_r}(a)$, we obtain

$$\lambda_r\frac{\partial R_{m,\,n}}{\partial s_{\lambda_r}(a)}=(-1)s_{\lambda_r}(b)\,\Sigma\frac{(-1)^{\rho_1+\rho_2+\cdots\cdots(\rho_r-1)+\cdots\cdots}}{\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\lambda_r{}^{\rho_r-1}\cdots\cdot\rho_1!\,\rho_2!\cdots\cdot(\rho_r-1)!\cdots\cdot}$$

$$(s_{\lambda_1}(a)s_{\lambda_1}(b))^{\rho_1}\cdots\cdot(s_{\lambda_r}(a)s_{\lambda_r}(b))^{\rho_r-1}\cdots\cdots \quad (13)$$

with $\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot\leq mn-\lambda_r$, and hence our equation is

$$\lambda_r\frac{\partial R_{m,\,n}}{\partial s_{\lambda_r}(a)}=(-1)s_{\lambda_r}(b)R_{m',\,n'},\ \text{with}\ m'n'=mn-\lambda_r.$$

By repeating this process until we have differentiated ρ_r times we get

$$\frac{\lambda_r{}^{\rho_r}\partial^{\rho_r}R_{m,\,n}}{\partial s_{\lambda_r}{}^{\rho_r}(a)}=(-1)^{\rho_r}s_{\lambda_r}{}^{\rho_r}(b)R_{m',\,n'}\cdots\cdots(14)$$

where $m',\ n'=mn-\lambda_r\rho_r$, and in a similar way we obtain more generally still,

$$(-1)^{\rho_1+\rho_2+\cdots\cdots}\lambda_1{}^{\rho_1}\lambda_2{}^{\rho_2}\cdots\cdot\frac{\partial^{\rho_1+\rho_2+\cdots\cdots}R_{m,\,n}}{\partial s_{\lambda_1}{}^{\rho_1}\partial s_{\lambda_2}{}^{\rho_2}\cdots\cdots}=s_{\lambda_1}{}^{\rho_1}s_{\lambda_2}{}^{\rho_2}\cdots\cdot R_{m',\,n'}\cdots\cdots \quad (15).$$

$$m'n'=mn-(\lambda_1\rho_1+\lambda_2\rho_2+\cdots\cdot)\cdots\cdots \quad (16)$$

where the differentiation may be distributed between the $s_\lambda(a)$'s and the $s_\lambda(b)$'s, and $R_{m',\,n'}$ may under certain circumstances be a resultant.

Examples: 1. $\dfrac{\partial R_{3,\,2}}{\partial s_2(a)}=-\tfrac{1}{2}s_2(b)R_{r,\,r'}$, where $rr'=4$, but $R_{r,\,r'}$ is neither the resultant $R_{4,\,1}$, nor $R_{2,\,2}$; the presence of terms $s_\lambda(a)$ where $\lambda>1$, which must contain b_2 prevents the former; the presence of terms $s_\lambda(a)$ where $\lambda>2$, which must contain a_3 prevents the latter,

2. $\dfrac{\partial R_{6,\,1}}{\partial s_5(a)}=-\tfrac{1}{5}s_5(b)R_{r,\,r'}$, where $rr'=1$, and here $R_{r,\,r'}=R_{1,\,1}$, is the resultant of $x+a_1$, and $1+b_1x$.

§ 10. DIFFERENTIATION OF THE TRANSCENDENTAL FORM.

The same differentiation applied to the transcendental form (10), or what is the same thing (as is seen by performing the differentiation in both cases) to the exponential form (8) is questionable. By (8) we have $R=e^{-\Sigma(1/r)s_r(a)s_r(b)}$. Differentiating with respect to $s_\lambda(a)$,

$$\frac{\lambda\partial R}{\partial s_\lambda(a)}=(-1)e^{-\Sigma(1/r)s_r(a)s_r(b)}s_\lambda(b),\text{ or }\lambda\frac{\partial R}{\partial s_\lambda(a)}=(-1)s_\lambda(b)R\quad\ldots\ldots(17),$$

with a corresponding formula

$$(-1)^{\rho_1+\rho_2+\cdots\cdots}\lambda_1^{\rho_1}\lambda_2^{\rho_2}\cdots\cdots\frac{\partial^{\rho_1+\rho_2+\cdots\cdots}R}{\partial s_{\lambda_1}{}^{\rho_1}\partial s_{\lambda_2}{}^{\rho_2}\cdots\cdots}=s_{\lambda_1}{}^{\rho_1}s_{\lambda_2}{}^{\rho_2}\cdots\cdots R\ldots\ldots\quad(18)$$

for the repitition. If (18) is allowed to stand it must be with the understanding that it is the universal type of all differential equations of the kind, that the R involved is the transcendental form, and that in any particular case such values in finite form must be picked out from the infinity of values which R admits, as will throw the indefinite form of (18) into that of (15) by which (18), if at all allowed, must be interpreted and explained.

II. APPLICATION TO SYMMETRIC FUNCTIONS.

§ 11. THE GENERAL TERM IN THE EXPANSION OF $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\ldots\alpha_\nu{}^{\kappa_\nu}$.

1. *In Terms of the s's.*

In terms of symmetric functions, the resultant $R_{m,n}$ of f and ϕ consists of terms of the form $b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\nu}\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\ldots\alpha_\nu{}^{\kappa_\nu}$. On the other hand we may express $R_{m,n}$ by (11) as a sum of terms of the form

$$\frac{(-1)^{\rho_1+\rho_2\cdots\cdots\rho_\kappa}}{\lambda_1\lambda_2\ldots\ldots\lambda_\mu\rho_1!\,\rho_2!\ldots\ldots\rho_\kappa!}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots\ldots s_{\lambda_\mu}(a)s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots\ldots s_{\lambda_\mu}(b)\ldots\ldots\quad(19)$$

by slightly changing the notation, ρ_1 being the number of λ's equal to λ_1, ρ_2 the number equal to λ_2, etc, and the summation to extend over all values of the λ's for which $\lambda_1+\lambda_2+\ldots\ldots\lambda_\mu\leqq mn$. In the term of $R_{m,n}$, as so expressed, we pick out the portion containing $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ from the product $s_{\lambda_1}(b)s_{\lambda_2}(b)\ldots\ldots s_{\lambda_\mu}(b)$, and obtain

$$(-1)^{\nu+\rho_1-\rho_2+\cdots\cdots}C_{\lambda,\kappa}b_{\kappa_1}b_{\kappa_2}\ldots\ldots b_{\kappa_\mu}s_{\lambda_1}(a)s_{\lambda_2}(a)\ldots\ldots s_{\lambda_\mu}(a)$$

where $\varkappa_1+\varkappa_2+\ldots\ldots\varkappa_\nu=\lambda_1+\lambda_2+\ldots\ldots\lambda_\mu$. By Waring's formula,

$$s_{\lambda_r}(b)=\lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots\ldots\tau_n}\frac{(\tau_1+\tau_2+\ldots\ldots\tau_n-1)!}{\tau_1!\,\tau_2!\ldots\ldots\tau_n!}b_1{}^{\tau_1}\,b_2{}^{\tau_2}\ldots\ldots b_n{}^{\tau_n}$$

with $\tau_1+2\tau_2+\ldots\ldots n\tau_n=\lambda_r$. Among the terms of $s_{\lambda_r}(b)$ is

$$\frac{\lambda_r(-1)^{\nu_r}(\nu_r-1)!}{\tau_1!\tau_2!\ldots\ldots}b_{\kappa_{r1}}\,b_{\kappa_{r2}}\ldots\ldots b_{\kappa_{r\nu}}$$

where $\varkappa_{r1}$, $\varkappa_{r2}\ldots\ldots$ form a set of numbers found among $\varkappa_1$, $\varkappa_2$, $\ldots\ldots\varkappa_\nu$, and τ_1

of the set are equal to $\varkappa_{r_1}$, τ_2 to $\varkappa_{r_2}$, etc., and where $\varkappa_{r_1}+\varkappa_{r_2}+\ldots\varkappa_{r_{\nu_r}}=\lambda_r$. In the same way from from $r=1$, to $r=\mu$, we obtain from μ sets with $\gamma_1+\gamma_2+\ldots\gamma_\mu=\nu$, similar factors, and therefore a term containing $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ in (19) is

$$\frac{(-1)^{\rho_1+\rho_2+\ldots\nu}}{\rho_1!\,\rho_2!\ldots\tau_1!\,\tau_2!\ldots}(\nu_1-1!)(\nu_2-1)!\ldots(\nu_\mu-1)!s_{\lambda_1}(a)\ldots s_{\lambda_\mu}(a)b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}\ldots(20)$$

In like manner other decompositions, if there be such, of the $\nu\varkappa$'s into μ sets, where the sum of the sets is equal to λ_1, $\lambda_2\ldots\lambda_\mu$ respectively, will give similar expressions, and the entire coefficient of $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}s_{\lambda_1}(a)\ldots s_{\lambda_\mu}(a)$ will be

$$\Sigma\frac{(-1)^{\rho_1+\rho_2+\ldots\rho_\kappa+\nu}}{\rho_1!\rho_2!\ldots\rho_\kappa!\,\tau_1!\tau_2!\tau_r!\ldots}(\nu_1-1)!(\nu_2-1)!\ldots(\nu_\mu-1)!=(-1)^{\rho_1+\rho_2+\ldots\nu}C_{\lambda,\,\kappa}\ldots(21).$$

Another term $s_{\mu_1}(a)s_{\mu_2}(a)\ldots s_{\mu_\kappa}(a)b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ will have a coefficient similarly formed, etc. Equating the terms in both developments of $R_{m,n}$ which contain $b_{\kappa_1}b_{\kappa_2}\ldots b_{\kappa_\nu}$ and dividing out this product, we have,

$$\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}=\Sigma(-1)^{\nu+\rho_1+\rho_2+\ldots\rho_\kappa}C_{\lambda,\,\kappa}S_\lambda(a)\ldots\ldots(22).$$

It will be seen that this formula agrees with that given in Faà di Bruno's Binäre Formen, page 8, except that the terms are grouped differently in the summation, formula (22) grouping the terms with respect to the numerical values of the subscripts of the s's, the latter grouping them with respect to the number of elements entering into a combination to form the subscript. The coefficient $(-1)^{\nu+\rho_1+\rho_2+\ldots}C_{\lambda,\,\kappa}$ of (21) is the coefficient of the general term $S_\lambda(a)$ in the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}$ in terms of the s's.

2. *In Terms of the a's.*

By using the preceding formula (22), we easily obtain a formula for the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}$ in terms of the a's. We have

$$s_{\lambda_r}(a)=\lambda_r\Sigma(-1)^{\tau_1+\tau_2+\ldots\tau_m}\frac{(\tau_1+\tau_2+\ldots\tau_m-1)!}{\tau_1!\,\tau_2!\ldots\tau_m!}a_1^{\tau_1}a_2^{\tau_2}\ldots a_m^{\tau_m}$$

with $\tau_1+2\tau_2+\ldots m\tau_m=\lambda_r$. Among the terms of $s_{\lambda_r}(a)$ is

$$\frac{(-1)^{\sigma_r}\lambda_r(\sigma_r-1)!}{\varkappa_1!\varkappa_2!\ldots\varkappa_\rho!}a_{\mu_{r_1}}a_{\mu_{r_2}}\ldots a_{\mu_{r\sigma_r}}$$

where $\mu_{r_1}+\mu_{r_2}+\ldots\mu_{r\sigma_r}=\lambda_r$, the $\varkappa$'s are for repetitions in the a's, $\sigma_1+\sigma_2+\ldots\sigma_\mu=\sigma$, and μ_{r_1}, $\mu_{r_2}\ldots$ are found among μ_1, μ_2, $\ldots\mu_\sigma$; similarly for other factors. Then a term in $(-1)^{\nu+\rho_1+\rho_2+\ldots}C_{\lambda,\,\kappa}S_\lambda(a)$ containing $a_{\mu_1}a_{\mu_2}\ldots a_{\mu_\sigma}$ is

$$(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots}\frac{\lambda_1\lambda_2\ldots}{\varkappa_1!\varkappa_2!\ldots}C_{\lambda,\,\kappa}(\sigma_1-1)!(\sigma_2-1)!\ldots(\sigma_\mu-1)!a_{\mu_1}a_{\mu_2}\ldots a_{\mu_\sigma}\ldots\ldots(23).$$

The sum of all numerical coefficients of the form contained in (23) is the complete coefficient of $a_{\mu_1}a_{\mu_2}\ldots a_{\mu_\sigma}$ in the expansion of $\Sigma\alpha_1{}^{\kappa_1}\alpha_2{}^{\kappa_2}\ldots\alpha_\nu{}^{\kappa_\nu}$ and this in his notation Gordan expresses as

$$\Sigma(-1)^{\sigma+\nu+\rho_1+\rho_2+\ldots}\frac{g_\lambda}{\tau_1!\,\tau_2!\ldots}(\varkappa_{11}\varkappa_{12}\ldots\mid\varkappa_{21}\varkappa_{22}\ldots\mid\ldots)$$
$$(\mu_{11}\mu_{12}\ldots\mid\mu_{21}\mu_{22}\ldots\mid\ldots)\ldots\ldots(24).$$

By means of this formula the general coefficient in the expansion of the binary resultant in terms of the a's and b's can of course be independently expressed.

Elmira, New York, December 19, 1899.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

[Continued from December Number.]

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\surd[1-e^2\sin^2\theta]} = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\surd[1-e^2\sin^2\theta]} - \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\surd[1-e^2\sin^2\theta]}.$$

$\therefore$ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\surd[1-e^2\sin^2\theta]} = \frac{1}{105e^8}[(48-128e^2+103e^4-15e^6)E(e, \tfrac{1}{2}\pi)$$

$$-4(12-38e^2+41e^4-15e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (20).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\surd[1-e^2\sin^2\theta]} = \int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\surd[1-e^2\sin^2\theta]} - \int_0^{\frac{1}{2}\pi} \frac{\cos^8\theta d\theta}{\surd[1-e^2\sin^2\theta]}.$$

$\therefore$ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \frac{\cos^8\theta d\theta}{\surd[1-e^2\sin^2\theta]} = \frac{1}{105e^8}[(48-208e^2+353e^4-298e^6+105e^8)F(e, \tfrac{1}{2}\pi)$$

$$-2(24-92e^2+132e^4-88e^6)E(e, \tfrac{1}{2}\pi)] \ \ldots. \ (21).$$

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\sin^6\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta d\theta}{\surd[1-e^2\sin^2\theta]} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta d\theta}{\surd[1-e^2\sin^2\theta]}.$$

$\therefore$ From (10) and (17) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\sin^6\theta d\theta = \frac{1}{105e^6}[(8+5e^2+11e^4-24e^6)F(e, \tfrac{1}{2}\pi)$$

$$-(8+9e^6+16e^4-48e^6)E(e, \tfrac{1}{2}\pi)] \ldots.. \ (22).$$

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\sin^4\theta\cos^2\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^2\theta d\theta}{\surd[1-e^2\sin^2\theta]} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^2\theta d\theta}{\surd[1-e^2\sin^2\theta]}$$

$\therefore$ From (11) and (18) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \surd(1-e^2\sin^2\theta)\sin^4\theta\cos^2\theta d\theta = \frac{1}{105e^6}[(8-5e^2-5e^4+8e^6)E(e, \tfrac{1}{2}\pi)$$

$$-(8-9e^2-3e^4+4e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (23).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^4\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}$$

∴ From (12) and (19) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\sin^2\theta\cos^4\theta d\theta = \frac{1}{105e^6}[(3-23e^2+18e^4-3e^6)F(e, \tfrac{1}{2}\pi)$$

$$-(8-19e^2+9e^4-6e^6)E(e, \tfrac{1}{2}\pi)] \quad \ldots\ldots \quad (24).$$

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta = \int_0^{\frac{1}{2}\pi} \frac{\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} - e^2\int_0^{\frac{1}{2}\pi} \frac{\sin^2\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}.$$

∴ From (13) and (20) by substitution we get,

$$\int_0^{\frac{1}{2}\pi} \sqrt{(1-e^2\sin^2\theta)}\cos^6\theta d\theta = \frac{1}{105e^6}[(8-32e^2+58e^4+15e^6)E(e, \tfrac{1}{2}\pi)$$

$$-(8-32e^2+74e^4-45e^6)F(e, \tfrac{1}{2}\pi)] \ldots\ldots \quad (25).$$

Let $n=3$ in (10_0), then $A_5 = \frac{8}{9}.\frac{1+e^2}{e}.A_4 - \frac{7}{9}A_3$.

$$\therefore A_5 = \frac{1}{\pi}\int_0^{2\pi} \frac{\cos 5\varphi d\varphi}{[1+e^2-2e\cos\varphi]^{\frac{1}{2}}}$$

$$= \frac{4}{315\pi e^5}[(128+40e^2+39e^4+44e^6+64e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots (19_0).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128+40e^2+39e^4+44e^6+64e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128+104e^2+99e^4+104e^6+128e^8)E(e, \tfrac{1}{2}\pi)] \quad \ldots\ldots (26).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^8\theta\cos^2\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-40e^2-21e^4-16e^6-16e^8)E(e, \tfrac{1}{2}\pi)$$

$$-(128-104e^2-9e^4-7e^6-8e^8)F(e, \tfrac{1}{2}\pi)] \ldots\ldots (27).$$

$$\int_0^{\frac{1}{2}\pi} \frac{\sin^6\theta\cos^4\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}} = \frac{1}{315e^{10}}[(128-248e^2+111e^4+5e^6+4e^8)F(e, \tfrac{1}{2}\pi)$$

$$-(128-184e^2+27e^4+11e^6+8e^8)E(e, \tfrac{1}{2}\pi)] \ldots\ldots \quad (28).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^6\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128-328e^2+243e^4-25e^6-10e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(128-392e^2+399e^4-130e^6-5e^8)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(29).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^8\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128-536e^2+855e^4-622e^6+175e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(128-472e^2+627e^4-334e^6+35e^8)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(30).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^{10}\theta d\theta}{\sqrt{[1-e^2\sin^2\theta]}}=\frac{1}{315e^{10}}[(128-616e^2+1179e^4-1126e^6+553e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(128-680e^2+1479e^4-1691e^6+1069e^8-315e^{10})F(c,\ \tfrac{1}{2}\pi)]\ldots\ldots(31).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{[1-e^2\sin^2\theta]}\sin^8\theta d\theta=\frac{1}{315e^8}[(16+8e^2+12e^4+28e^6-64e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16+16e^2+21e^4+40e^6-128e^8)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(32).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{[1-e^2\sin^2\theta]}\sin^6\theta\cos^2\theta d\theta=\frac{1}{315e^8}[2(8-4e^2-3e^4-4e^6+8e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(16-16e^2-3e^4-5e^6+8e^8)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(33).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{[1-e^2\sin^2\theta]}\sin^4\theta\cos^4\theta d\theta=\frac{1}{315e^8}[4(4-10e^2+6e^4+e^6-e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16-32e^2+9e^4+7e^6-8e^8)E(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots(34).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{[1-e^2\sin^2\theta]}\sin^2\theta\cos^6\theta d\theta=\frac{1}{315e^8}[2(8-28e^2+33e^4-10e^6+5e^8)E(e,\ \tfrac{1}{2}\pi)$$

$$-(16-64e^2+93e^4-50e^6+5e^8)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(35).$$

$$\int_0^{\frac{1}{2}\pi}\sqrt{[1-e^2\sin^2\theta]}\cos^8\theta d\theta=\frac{1}{315e^8}[4(4-22e^2+51e^4-68e^6+35e^8)F(e,\ \tfrac{1}{2}\pi)$$

$$-(16-80e^2+165e^4-194e^6-35e^8)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots(36).$$

Thus we can proceed as far as we desire, finding the values of similar expressions almost without number.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

II. Solution by J. M. ARNOLD, Crompton, R. I.

In regard to the solution of Problem 120, given in the January number, it seems to me that the total number of balls, 2147, is not the maximum number that can be placed in the box.

Suppose we pack it this way: Place 144 balls on the bottom, then a layer of 121, and so on alternately until fifteen layers have been put in.

Now fill with sawdust so that the sawdust is just level with the tops of the balls. We have left a clear space of 12×12 and a little more than 1 inch in depth. Into this space can be placed 150 balls, by first placing a row of 12 balls against one side of the box, then a row of 11, and so on alternately until thirteen rows are put in; making 7 rows of 12 and 6 of 11 balls each, 150 in all.

The distance between the center lines of the rows will be $\frac{1}{2}\sqrt{3}$.

And $12\times\frac{1}{2}\sqrt{3}+1=11.3923$. $12-11.3923=.6077$ inch to spare.

For the whole box we have:

Eight layers of 144 each	=1152
Seven layers of 121 each	= 847
One layer of	150
Total	=2149

124. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

At what time between 5 and 6 o'clock is the minute hand midway between 12 and the hour hand? When is the hour hand midway between 4 and the minute hand?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Take a and $a+1$ as the hours between which the conditions are to happen.

Let x=the distance, after a o'clock, the minute hand must move.

Let $\frac{1}{12}x$=the distance the hour hand moves.

(1). The first question of the problem involves two positions.

(a). In the first position, we readily find $5a+\frac{1}{12}x=2x$.

$$\therefore x=\frac{60a}{23}.$$

(b). In the second position we find $5(12-a)-\frac{1}{12}x=2(60-x)$.

$$\therefore x=\frac{60(12+a)}{23}.$$

Now put $a=5$. Then, in the first position, $x=13\frac{1}{23}$; and the time is $13\frac{1}{23}$ minutes past 5 o'clock.

In the second position, $x=44\frac{8}{23}$; and the time is $44\frac{8}{23}$ minutes past 5 o'clock.

(2). In the second question of the problem, take $4=b$.

Then, in general terms, we find

$$\frac{\pm(x-5b)}{2}=\pm[5(a-b)+\tfrac{1}{12}x].$$

$\therefore x=6(2a-b)$.

Plus of the double sign is to be used for $b<a$, and *minus* for $b>a$.

Put $a=5$, and $b=4$.

Then $x=36$, and the time is 36 minutes past 5 o'clock.

II. Solution by J. OWEN MAHONEY, B. E., M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and JENNIE B. BRIDENBAUGH, Student, Nebraska State Normal School, Peru, Neb.

Let x=the number of minute spaces over which the minute hand passes in each case. Then, in the first case, it is easily seen that

$$x+\tfrac{11}{12}x=25, \text{ or } x=13\tfrac{1}{23}.$$

Hence, at $13\frac{1}{23}$ minutes past 5 o'clock the first condition is satisfied.

In the second case,

$$x-2(\tfrac{1}{12}x+5)=20, \text{ or } x=36.$$

Hence, at 36 minutes past 5 o'clock the second condition is satisfied.

III. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.; and J. T. FAIRCHILD, Principal, Crawfis College, Ohio.

A. 1. $\frac{2}{2}$=distance hour-hand moves past 5.

2. $\frac{24}{2}$=distance minute-hand moves in the same time.

3. $\frac{2}{2}+25$ minutes=distance from 12 to the hour-hand.

4. $\frac{1}{2}$ of ($\frac{2}{2}+25$ minutes)$=\frac{1}{2}+12\frac{1}{2}$ minutes=distance from 12 to the minute-hand, since it is to be half way between 12 and the hour-hand.

5. $\therefore \frac{24}{2}=\frac{1}{2}+12\frac{1}{2}$ minutes.

6. $\therefore \frac{23}{2}=12\frac{1}{2}$ minutes, and

7. $\frac{1}{2}=\frac{1}{23}$ of $12\frac{1}{2}$ minutes$=\frac{25}{46}$ minutes, and

8. $\frac{24}{2}=24$ times $\frac{25}{46}$ minutes$=13\frac{1}{23}$ minutes, the time past 5, when the minute-hand will be midway between 12 and the hour-hand.

B. 1. $\frac{2}{2}$=distance hour-hand moves past 5.

2. $\frac{2}{2}+5$ minutes=distance from 4 to hour-hand.

3. $\frac{x}{2}+10$ minutes=distance from 4 to minute hand, since hour-hand is half way between 4 and the minute-hand.

4. $\frac{x}{2}+30$ minutes=distance from 12 to minute hand.

5. ∴ $\frac{24}{2}=\frac{x}{2}+30$ minutes.

6. ∴ $\frac{20}{2}=30$ minutes.

7. $\frac{1}{2}=\frac{1}{20}$ of 30 minutes=$1\frac{1}{2}$ minutes.

8. $\frac{24}{2}=24$ times $1\frac{1}{2}$ minutes=36 minutes=time past 5.

IV. Solution by Hon. JOSIAH H. DRUMMOND, Portland, Me.; ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Ia.; W. MANZILLA, Instructor in Mathematics, Langston University, Langston, Okla., and JOHN M. COLAW, A. M., Monterey, Va.

As the minute-hand travels over twelve minute-spaces while the hour-hand is going over one, the distance from XII to the hour-hand is $25+\frac{1}{12}$ of the distance the minute-hand travels ; and one-half of this is $\frac{25}{2}+\frac{1}{24}$ of that distance.

Hence, $\frac{25}{2}$ is $\frac{23}{24}$ of that distance which is $13\frac{1}{23}$, and the time is $13\frac{1}{23}$ minutes past five.

Second. Of course at two o'clock the hour-hand is precisely half-way between IV and the minute-hand.

Again, while the minute-hand travels from XII to IV, the hour-hand goes over $\frac{20}{12}$ minute-spaces. Hence $\frac{20}{12}+\frac{1}{12}$ of the distance the minute-hand travels after it reaches IV=one-half of that distance.

Hence $\frac{20}{12}$ is $\frac{5}{12}$ of that distance which is four. And the time required is twenty-four minutes past four.

ALGEBRA.

99. Proposed by CHARLES HALLETTE JUDSON, Professor of Mathematics, Furman University, Greenville, S. C.

Seven persons met at a summer resort, and agreed to remain as many days as there are ways of sitting at a round table, so that no one shall sit twice between the same two companions. They remained fifteen days. It is required to show in what way they may have been seated.

I. Solution by P. H. PHILBRICK, C. E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles, La.

Each person has six companions ; and the the number of ways of sitting are, therefore, equal to the number of ways of selecting two out of six or 6.5/1.2 =15 ways.

II. Solution by the PROPOSER.

I give below the seating of 6 persons 10 ways, and of 8 persons 21 ways, but I have failed to get a correct seating of 7.

n persons can be seated in $\dfrac{(n-1)(n-2)}{2}$ ways.

1	A^1	B^2	C^3	D^4	E^5	F^6
2	A^1	B^2	D^4	C^3	F^6	E^5
3	A^1	B^2	E^5	F^6	D^4	C^3
4	A^1	B^2	F^6	E^5	C^3	D^4
5	A	C	F	D	B	E
6	A	C	E	B	F	D
7	A	C	B	E	D	F
8	A	D	E	C	B	F
9	A	D	B	F	C	E
10	A	E	D	B	C	F

1	A	B	C	D	E	F	G	H
2	A	B	D	F	G	C	H	E
3	A	B	E	G	F	H	C	D
4	A	B	F	C	H	D	E	G
5	A	B	G	H	C	E	D	F
6	A	B	H	E	D	G	F	C
7	A	C	G	E	H	F	B	D
8	A	C	D	H	E	B	F	G
9	A	C	E	F	B	G	D	H
10	A	C	H	B	F	D	G	E
11	A	C	B	D	G	H	E	F
12	A	D	H	G	E	C	B	F
13	A	D	F	E	G	B	C	H
14	A	D	G	C	B	H	F	E
15	A	D	E	B	C	F	H	G
16	A	E	C	F	D	H	B	G
17	A	E	D	B	H	G	C	F
18	A	E	B	G	C	D	F	H
19	A	F	H	D	B	E	C	G
20	A	F	G	B	D	C	E	H
21	A	G	D	C	F	E	B	H

In seating the 8 persons, the first six arrangements are quite simple, as indicated. Now take 6 reversed, and permute them just as the first 6. This gives the next 5. Take any other, as 3, reversed, and permute in same way. Take 4 in same way rejecting first 1, then 2, then 3, then 4, which have been previously arranged, and we finally get 21.

[Neither of the above solutions show *how* to seat 7 persons in the 15 ways. For the first correct arrangement sent us, we are authorized to offer a year's subscription to the MONTHLY. EDITOR.]

100. **Proposed by W. H. CARTER, Vice President, and Professor of Mathemetics, Centenary College, Jackson, La.**

Solve, $x^{x-y}=y^{4a}$, $y^{x+y}=x^a$.

I. **Solution by CHARLES C. CROSS, Meredithville, Va.; P. S. BERG, A. M., Larimore, N. D.; J. D. CRAIG, Frankfort, N. J.; JOSIAH H. DRUMMOND, LL. D.. Portland, Me.; ALOIS F. KOVARIK, Decorah, Ia.; P. H. PHILBRICK, C. E., Lake Charles, La.; and COOPER D. SCHMITT, A. M., Knoxville, Tenn.**

Taking logarithms, we have, $(x+y)\log x=4a\log y$......(1);

and $(x+y)\log y=a\log x$ .....(2).

(1) divided by (2) gives $\log^2 x=4\log^2 y$,

or $\log x=2\log y$......(3);

Therefore $x=y^2 \ldots\ldots(4)$.

Substituting (3) in (2), and dividing by $\log y$,

we get $x+y=2a \ldots\ldots(5)$.

Substituting (4) in (5), $y^2+y=2a$.

Therefore, $y=\frac{1}{2}[-1\pm\surd(8a+1)]$.

$\therefore x=\frac{1}{2}[4a+1\mp\surd(8a+1)]$.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

$$(x+y)\log x=4a\log y \ldots\ldots (1);$$

$$(x+y)\log y=a\log x \ldots\ldots (2).$$

(1)$\div$(2) gives $(\log x)^2=4(\log y)^2$.

$\therefore \log x=\pm 2\log y$. $\therefore x=y^2$ or $1/y^2$.

$\therefore 2(y^2+y)=4a$. $\therefore y^2+y=2a$.

$\therefore y=-\frac{1}{2}[1\pm\surd(1+8a)]$; $x=\frac{1}{2}[1+4a\mp\surd(1+8a)]$.

When $x=1/y^2$, we get $y^3+2ay^2=-1$.

The roots of this equation can be found by Cardan's Rule.

[ZERR, SCHEFFER]

Rational values are obtained by substituting for a the triangular numbers 1, 3, 6, 10, 15, 21, 28, etc. The equation $y^{-2}+y+2a=0$ leads to the cubic $y^3+2ay^2+1=0$. In case of $a=3$, we have the simultaneous sets

$$y=2\ ;\ y=-3\ ;\ y=-6.0275\ ;$$

$$x=4\ ;\ x=9\ ;\ x=-.0275\ ;$$

and two imaginary roots. [SCHEFFER.]

III. Solution by WALTER HUGH DRANE, Graduate Student, Harvard University, Cambridge, Mass.; and CHAS. E. MYERS, Canton, Ohio.

$x^{x+y}=y^{4a} \ldots\ldots (1)$. $y^{x+y}=x^a \ldots\ldots(2)$.

From (1) $y=x^{(x+y)/4a}$, and from (2) $y=x^{a/(x+y)}$.

$\therefore x^{(x+y)/4a}=x^{a/(x+y)} \ldots\ldots(3)$. $\therefore (x+y)/4a=a/(x+y) \ldots\ldots (4)$.

$\therefore x+y=\pm 2a \ldots\ldots(5)$.

This in (1) gives, taking $+$ value of a, $x^{2a}=y^{4a}$.

$\therefore x=y^2$. $\therefore$ By (2) $y^{2+y}=y^{2a}$. $\therefore y^2+y=2a$.

$$\therefore y=\frac{-1\pm\surd(1+8a)}{2};\ \therefore x=\frac{1+4a\mp\surd(1+8a)}{2}.$$

[DRANE, MYERS.]

Taking the negative value of $2a$ leads us to the cubic $y^3+2ay^2+1=0$, and $x=1/y^2$. Substitute $z=y+2a/3$, and this reduces to $z^3-4a^2z+1=0$. Solving this by Cardan's method gives

$$u=\tfrac{1}{2}[-1\pm\surd(1-\tfrac{256}{27}a^6)],\ \text{and}\ v=\tfrac{1}{2}[-1\mp\surd(1-\tfrac{256}{27}a^6)],$$

where $u^{\frac{1}{3}}+v^{\frac{1}{3}}=z$, and $u^{\frac{1}{3}}v^{\frac{1}{3}}=4a^2/3$. $\therefore$ Three values of z are,

$$z_1=\left(\frac{-1\pm\sqrt{(1-\frac{256}{27}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{[-1\pm\sqrt{(1-\frac{256}{27}a^6)}]^{\frac{1}{3}}};$$

$$z_2=w\left(\frac{-1\pm\sqrt{(1-\frac{256}{27}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{w[-1\pm\sqrt{(1-\frac{256}{27}a^6)}]^{\frac{1}{3}}};$$

$$z_3=w^2\left(\frac{-1\pm\sqrt{(1-\frac{256}{27}a^6)}}{2}\right)^{\frac{1}{3}}+\frac{4a^2\sqrt[3]{2}}{w^2[-1\pm\sqrt{(1-\frac{256}{27}a^6)}]^{\frac{1}{3}}}.$$

$\therefore$ $y_1=z_1-2a/3$, $y_2=z_2-2a/3$, and $y_3=z_3-2a/3$, are the three required values of y; and $x_1=1/y_1^2$, $x_2=1/y_2^2$, $x_3=1/y_3^2$, are the three values of x. It is easily seen that the values of y, and hence of x, are all real, one real and two imaginary, or real and multiple, according as $a^6>$, $<$, or $=\frac{27}{256}$. [DRANE.]

GEOMETRY.

122. Proposed by G. I. HOPKINS, A. M., Professor of Mathematics and Physics, Manchester High School, Manchester, N. H.

If perpendiculars are dropped from the vertices of a regular polygon upon any diameter of the circumscribed circle, the sum of the perpendiculars which fall on one side of this diameter is equal to the sum of those which fall on the opposite side. [From Chauvenet's *Treatise on Elementary Geometry*.]

III. Solution by WILLIAM H. ECHOLS, B. S., C. E., Professor of Mathematics in University of Virginia, Charlottesville, Va.

A very simple proof of this is:

1°. Let there be a regular polygon of $2n$ sides, p_1, ...p_{2n}, the perpendiculars from vertices on any straight line.

Then if p_c is perpendicular from the center, clearly

$$\sum_1^{2n} p_r=2np_c.$$

2°. Form a polygon of n sides by joining the mid-points of alternate sides of 1°. Then at once

$$2\sum_1^{n} p'_r=\sum_1^{2n} p_r=2np_c,$$

where p'_r are perpendiculars of n-side.

This is true whatever integer be n.

3°. If the line of projection passes through the center, the theorem is demonstrated generally.

125. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

To find the locus of a point on the surface of an ellipsoid which has the property that the tangent plane at that point is at the given distance, f, from the center of the ellipsoid.

Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The equation to the tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\ldots\ldots(1)$$

at the point (x', y', z') is

$$\frac{x'x}{a^2}+\frac{y'y}{b^2}+\frac{z'z}{c^2}=1\ \ldots\ldots(2).$$

Then

$$\frac{1}{f^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}\ldots\ldots(3),$$

is the required locus, an ellipsoid concentric with (1).

Also solved by *G. B. M. ZERR*, and *ELMER SCHUYLER*.

126. Proposed by GEORGE R. DEAN, Professor of Mathematics, University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Through any fixed point O draw two straight lines at right angles. Let one line cut a given circle at Q, the other at R. Find, by Euclidean methods, the locus of the foot of the perpendicular from O upon the chord QR. Give complete analysis and discussion. Solve also by coördinate geometry.

I. Solution (Analytical) by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Let r=the radius of the given circle, a=the distance of its center from O, and take the line through O and the center for the axis of x. Then the coördinate axes being rectangular, the equation to the fixed circle is

$$(x-a)^2+y^2=r^2 \text{ or } x^2+y^2-2ax+a^2-r^2=0\ldots\ldots(1).$$

If $lx+my=1$ .....(2) be the equation to QR, the equation to the lines OQ and OR is

$$x^2+y^2-2ax(lx+my)+(a^2-r^2)(lx+my)^2=0\ldots\ldots(3), \text{ or}$$

$$[1-2al+(a^2-r^2)l^2]x^2+2[lm(a^2-r^2)-am]xy+[1+(a^2-r^2)m^2]y^2=0\ \ldots\ldots(4).$$

The condition that these lines are perpendicular to each other is

$$2-2al+(a^2-r^2)(l^2+m^2)=0\ldots\ldots(5).$$

The line through O and perpendicular to (2) is $x/y=l/m$......(6).

Making (5) homogeneous by aid of (2), we have,

$$[2x^2-2ax+(a^2-r^2)]l^2+(4xy-2ay)lm+[2y^2+(a^2-r^2)]m^2=0\ldots\ldots(7).$$

l and m by (6) being proportional to x and y, (7) easily becomes

$$2x^4-2ax^3+(a^2-r^2)x^2+4x^2y^2-2axy^2+2y^4+(a^2-r^2)y^2=0, \text{ or}$$

$$2(x^2+y^2)^2-[2ax-(a^2-r^2)](x^2+y^2)=0 \ldots\ldots (8),$$

giving the two circles

$$x^2+y^2=0 \ldots\ldots (9),\ x^2+y^2-ax+\tfrac{1}{2}(a^2-r^2)=0 \ \ldots\ldots (10)$$

for the required loci.

II. **Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; and J. SCHEFFER, A. M., Hagerstown, Md.**

To solve the problem by coördinate geometry, we may proceed as follows: Let the fixed point be the origin, and let the axis of x pass through the center of the circle. Its equation is

$$x^2+y^2+2gx+c=0 \ \ldots\ldots (1).$$

Let the chord, QR, be $y=mx+a \ldots\ldots (2)$.

Make (1) homogeneous by means of the relatiou $(y-mx)/a=1$. The resulting equation, viz.,

$$x^2+y^2+2gx\frac{(y-mx)}{a}+\frac{c(y-mx)^2}{a^2}=0 \ \ldots\ldots (3),$$

represents the lines through O, and Q and R.

(3) may be written $(a^2-2agm+cm^2)x^2+(a^2+c)y^2+2(ag-cm)xy=0$.

But the lines OQ and OR are at right angles. The condition for this is

$$(a^2-2agm+cm^2)+(a^2+c)=0, \text{ or } 2a^2+cm^2-2agm+c=0 \ldots\ldots (4).$$

The perpendicular from O and QR is $y=-(1/m)x \ldots\ldots (5)$.

We must find the locus of the intersection of (2) and (5), under the condition (4). (5) gives $m=-x/y$. (2) gives $a=y-mx=(y^2+x^2)/y$.

Substitute for m and a in (4), and we have

$$2\frac{(y^2+x^2)^2}{y^2}+c\left(\frac{x^2}{y^2}+1\right)+2g\frac{(x^2+y^2)x}{y^2}=0, \text{ or } 2(x^2+y^2)+2gx+c=0,$$

the required locus. This is evidently a circle, with center half way between O and the center of the given circle.

NOTE ON PROBLEM 118.

BY GEORGE R. DEAN, ROLLA, MO.

The analysis of a geometrical problem has always appeared to me more interesting and useful than the mere construction and proof.

Any problem in maxima and minima may be solved by applying the general principle that a maximum or a minimum value of a variable quantity lies between two equal values. Let P be any point in the line FH. At some other point Q the angle DQC is equal to the angle DPC. The point Q is obviously found by drawing a circle through D, C, and P. The point at which the angle is a maximum is situated between P and Q. It is obvious that if the point P had been so selected that Q would coincide with it, the required point would be determined. A circle drawn through D and C tangent to the given line will cut the line in two coincident points between which the maximum point lies, and is therefore coincident with them.

As an example of this kind of analysis I propose the following problem :

Find a point in a given line such that the sum of its distances from two fixed points is a minimum. Give the analysis.

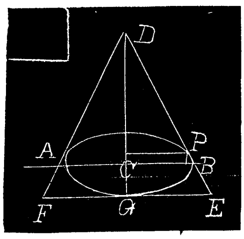
CALCULUS.

94. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Find the minimum isosceles triangle that can be described about a given ellipse, having its base parallel to the major axis.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; M. C. STEVENS, A. M., Professor of Higher Mathematics, Purdue University, Lafayette, Ind.; and H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia. Pa.

Let x, y be the coördinates of P. Then $x^2/a^2+y^2/b^2=1$ is the equation to the ellipse.



$CD=b^2/y$. $\therefore$ $GD=b^2/y+b=b(b+y)/y$.

$AC=a^2/x$. $\therefore$ $FG=a^2(b+y)/bx$.

$\therefore$ $a^2(b+y)^2/xy=$area$=$minimum.

$\therefore$ $dy/dx=y(y+b)/x(y-b)$.

Also $dy/dx=-b^2x/a^2y$, from the equation to the ellipse.

$$\therefore \frac{b^2x}{a^2y}=\frac{y(b+y)}{x(b-y)}, \text{ or } x^2=\frac{a^2y^2(b+y)}{b^2(b-y)}.$$

This value of x^2 in the equation to the ellipses gives, after reduction, $2y^2+by=b^2$.

$\therefore$ $y=\frac{1}{2}b$ for a minimum.

$\therefore$ Altitude$=b(b+y)/y=3b$. Base$=2a^2(y+b)/bx=3a^2/x=2a\sqrt{3}$.

Side$=\sqrt{(9b^2+3a^2)}$. Area$=3ab\sqrt{3}$.

III. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Let the equation to tangent DE be $(x/a)\cos\phi+(y/b)\sin\phi=1$.
Then $CB=a\sec\phi$; $CD=b\text{cosec}\phi$. $DG=b+b\text{cosec}\phi$.
Area$=DG.GE$. $GE : CB :: DG : DC$. Whence

$$GE=\frac{a\sec\phi(1+\text{cosec}\phi)}{\text{cosec}\phi}=\frac{a\sec\phi(1+\text{cosec}\phi)}{\text{cosec}\phi}b(1+\text{cosec}\phi)=ab\tan\phi(1+\text{cosec}\phi)^2$$

Hence $\tan\phi(1+\text{cosec}\phi)^2$ is to be examined for a minimum. Differentiating and equating to zero, we have

$$\sec^2\phi(1+\text{cosec}\phi)^2-2\tan\phi(1+\text{cosec}\phi)\text{cosec}\phi\cot\phi=0,$$

whence, $\sec^2\phi(1+\text{cosec}\phi)=2\text{cosec}\phi$, or

$$\frac{1+\sin\phi}{\sec\phi\cos^2\phi}=\frac{2}{\sin\phi}.$$

Solving, $\sin\phi=\frac{1}{2}$; whence $\text{cosec}\phi=2$.
Hence $DG=b+b\text{cosec}\phi=b+2b=3b$.

Also solved by *P. S. BERG*, *W. H. DRANE*, *J. SCHEFFER*, and *J. W. YOUNG*.

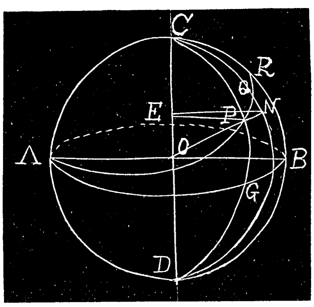
95. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

A ship starts at the equator and sails northeast at all times. How far has the ship sailed (in miles) when her latitude is 30°, 45°, 60°, 90°? How far when her longitude is 90°, 180°, 270°, 360°? Regarding the earth as a sphere, radius 3956 miles.

Solution by the PROPOSER.

Let A be the point of the ship's departure, $APQR$ the ship's course, P, Q two consecutive points on the course, $AG=\theta=$longitude of P, $PG=\phi=$latitude of P, $OP=r=$radius of the earth, $\angle PQN=\beta=\frac{1}{4}\pi$.

Then $PQ=ds$, $PN=EP\times\angle PEN=r\cos\phi d\theta$, $QN=rd\phi$.



$\therefore ds^2=r^2(\cos^2\phi d\theta^2+d\phi^2)$.

$\therefore ds=r(\cos^2\phi d\theta^2+d\phi^2)^{\frac{1}{2}}$ .....(1).

$PN/QN=\tan\beta=\cos\phi d\theta/d\phi$.

$\therefore d\theta=\tan\beta d\phi/\cos\phi$......(2).

(2) in (1) gives $ds=\sqrt{(1+\tan^2\beta)}d\phi=rd\phi/\cos\beta$.

$\therefore s=(r/\cos\beta)\int_0^\Phi d\phi=r\phi/\cos\beta=r\phi\sqrt{2}$.

When $\phi=\frac{1}{6}\pi$, $s=\frac{1}{6}\pi r\sqrt{2}=2929.3817$ miles.
When $\phi=\frac{1}{4}\pi$, $s=\frac{1}{4}\pi r\sqrt{2}=4394.07257$ miles.
When $\phi=\frac{1}{3}\pi$, $s=\frac{1}{3}\pi r\sqrt{2}=5858.7634$ miles.
When $\phi=\frac{1}{2}\pi$, $s=\frac{1}{2}\pi r\sqrt{2}=8788.14514$ miles.

From (2) $d\phi = \cos\phi d\theta/\tan\beta$......(3).
(3) in (1) gives

$$ds = r\cos\phi\sqrt{(1+\tan^2\beta)}d\theta/\tan\beta = r\cos\phi d\theta/\sin\beta.$$

From (2), $\theta = \tan\beta\int\frac{d\phi}{\cos\phi} = \tan\beta\log[\tan(\frac{1}{4}\pi+\frac{1}{2}\phi)]$.

$$\therefore\ e^{\theta\cot\beta} = \tan(\tfrac{1}{4}\pi+\tfrac{1}{2}\phi) = \frac{1+\sin\phi}{\cos\phi}.\quad \therefore\ \cos\phi = \frac{2}{e^{\theta\cot\beta}+e^{-\theta\cot\beta}} = \frac{2}{e^{\theta}+e^{-\theta}},$$

since $\cot\beta = 1$.

$$\therefore\ s = 2r\sqrt{2}\int_0^{\theta}\frac{d\theta}{e^{\theta}+e^{-\theta}} = 2r\sqrt{2}(\tan^{-1}e^{\theta}-\tfrac{1}{4}\pi).$$

When $\theta = \frac{1}{2}\pi$, $s = 2r\sqrt{2}(\tan^{-1}e^{\frac{1}{2}\pi}-\frac{1}{4}\pi) = r\sqrt{2}(.3695185\pi) = 6494.764423$ miles.
When $\theta = \pi$, $s = 2r\sqrt{2}(\tan^{-1}e^{\pi}-\frac{1}{4}\pi) = r\sqrt{2}(.472506\pi) = 8304.902620$ miles.
When $\theta = \frac{3}{2}\pi$, $s = 2r\sqrt{2}(\tan^{-1}e^{\frac{3}{2}\pi}-\frac{1}{4}\pi) = r\sqrt{2}(.494281\pi) = 8687.626341$ miles.
When $\theta = 2\pi$, $s = 2r\sqrt{2}(\tan^{-1}e^{2\pi}-\frac{1}{4}\pi) = r\sqrt{2}(.498804\pi) = 8767.1239018$ miles.

Also solved by *J. SCHEFFER*. A somewhat different solution of this problem is given in Finkel's *Mathematical Solution Book*, page 344.

96. Proposed by W. H. CARTER, Vice President, and Professor of Mathemetics, Centenary College, Jackson, La.

If $f(x) = \int f(x)dx$, find $f(x)$, the constant being zero.

Solution by W. F. SHAW, 1600 Sabine Street, Austin, Tex.

$f(x) = \int f(x)dx$. $\quad df(x) = f(x)dx$.

$\frac{df(x)}{f(x)} = dx$. $\quad \log f(x) = x$. $\quad f(x) = e^x$.

Also solved by *W. H. DRANE*, *J. SCHEFFER*, and *G. B. M. ZERR*.

MECHANICS.

87. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

"He on his impious foes onward drove,
Drove them before him to the bounds
And crystal walls of Heaven; which opening wide
Rolled inward and a spacious gap disclosed
Into the wasteful deep; headlong themselves they threw
Down from the verge of Heaven.
Nine days they fell; Hell at last
Yawning received them whole and on them closed!"

Paradise Lost, Book VI.

Assuming Hell to be the center of the earth, and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

$$\text{Nine days}=9\times24\times60\times60=777600 \text{ seconds}$$
$$\text{Deduct} \quad 475 \text{ seconds}$$

$\therefore$ Time of falling to surface of the earth $=777125$ seconds.

The corresponding distance is, $\frac{1}{2}gt^2=16\frac{1}{12}(777125)^2=9713099188802$ feet $=1839602119$ miles.

Also solved by *H. C. WHITAKER*, *G. B. M. ZERR*, *B. F. SINE*, *J. SCHEFFER*, *W. H. DRANE*, and *J. B. GREGG*. Mr. Gregg furnished a very elaborate solution, indicating the various steps of the computation, his result being 360733 miles. Dr. Whitaker gets the same result. The results of the various contributors differ widely, due to variously assumed values of the constants, and, in some cases, considering the earth a mere point.

95. Proposed by FLORIAN CAJORI, Ph. D., Author of History of Mathematics, History of Physics, etc., and Professor of Mathematics, Colorado College, Colorado Springs, Col.

Assuming that the velocity is proportionate to the distance described from the state of rest, (1) can the body start in motion? (2) If it can, what is its initial acceleration? If we make the additional assumption that the time of fall, from rest, through a finite distance is finite, does it follow that the distance is infinite?

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

1. The statement here is somewhat confusing. No body can start in motion unless acted upon by an external force; if it be meant here then to ask, can the body be started, the answer is self-evident. Theoretically, an infinitesimal force would be sufficient to put the body in motion though the time might become infinite before the velocity became finite.

2. We have $ds/dt=ks$ where k is a constant depending upon initial conditions. Differentiating we get $d^2s/dt^2=k(ds/dt)=k^2s$; that is, the initial acceleration is k time the initial velocity and is itself proportional to the distance. What the value of this initial acceleration is, depends upon the external force acting and the mass of the body.

3. Nothing is said here about the initial force acting. If we assume it infinitesimal, it follows from 1, that if the time of fall through a finite distance is finite, then the velocity must be infinite.

DIOPHANTINE ANALYSIS.

78. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics in University of Tennessee, Knoxville, Tenn.

Find three square numbers in harmonical progression.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The terms of an harmonical progression are the reciprocals of such numbers as form an arithmetical progression.

Let a^2, b^2, and c^2 be three square numbers in arithmetical progression, $a<b<c$. Then $b^2-a^2=c^2-b^2$, or $a^2+c^2=2b^2$. a, b, and c are rendered rational and integral by putting $a=p^2-q^2$ the difference between $2pq$, $b=p^2+q^2$, and $c=p^2-q^2+2pq$.

$\therefore$ The required harmonical progression is

$$1/(p^2-q^2-2pq)^2,\ 1/(p^2+q^2)^2,\ 1/(p^2-q^2+2pq)^2,$$

in which p and q are any integers, $p>q$.

Put $p=2$ and $q=1$. We then have $1/1^2$, $1/5^2$, $1/7^2$.

Put $p=3$ and $q=2$. We then have $1/7^2$, $1/13^2$, $1/17^2$.

II. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

If a^2, b^2 are two numbers, then will a^2, $2a^2b^2/(a^2+b^2)$, b^2 be in harmonical progression. Let $a=m-n$, $b=m+n$; therefore, $(m-n)^2$, $(m^2-n^2)^2/(m^2+n^2)$, $(m+n)^2$ will be the required numbers, if $m^2+n^2=\square$.

Let $m=p^2-q^2$, $n=2pq$.

$\therefore (q^2+2pq-p^2)^2$, $(p^4-6p^2q^2+q^4)^2/(p^2+q^2)^2$, $(p^2+2pq-q^2)^2$ are the numbers required.

Let $p=2$, $q=1$. $\therefore 1$, $\frac{49}{25}$, 49 are the numbers.

Let $a=m$, $b=7m$. Then m^2, $49m^2/25$, $49m^2$ are the numbers required.

Let $m=5n$. $\therefore 25n^2$, $49n^2$, $1225n^2$ are the required numbers. This gives a series of whole numbers by giving n integral values.

III. Solution by JOSIAH H. DRUMMOND, LL. D., Portland, Me.; A. H. BELL, Hillsboro, Ill.; and CHAS. C. CROSS, Meredithville, Va.

As the reciprocal of numbers in arithmetical progression are in harmonical progression, we may solve the question by obtaining three squares in arithmetical progression.

Let $x^2-2xy+y^2$, x^2+y^2, and $x^2+2xy+y^2$, be the three squares, and it only remains to make $x^2+y^2=\square$. This is so when $x=p^2-q^2$ and $y=2pq$; then $x^2+y^2=(p^2+q^2)^2$ in which p and q may be any unequal numbers.

Take $p=2$, and $q=1$, and the squares are 1, 25, and 49, and the reciprocals are 1, $\frac{1}{25}$, and $\frac{1}{49}$, or if integrals are desired, 1225, 49, and 25.

Also solved by *O. S. WESTCOTT*, *SYLVESTER ROBINS*, *ALOIS F. KOVARIK*, *J. W. YOUNG*, and the *PROPOSER*.

79. Proposed by EDMUND FISH, Hillsboro, Ill.

Find an integral right triangle in which the bisector of one of the acute angles is also integral.

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

In any right triangle, let a=altitude, b=base, c=hypotenuse, c_1=bisector of angle opposite base, c_2=bisector of angle opposite altitude.

Put $a=tpq$, $b=t(p^2-q^2)/2$, and $c=t(p^2+q^2)/2$; t, p, and q being any integers, $p>q$.

Then from solution I, problem 43, page 95, Vol. IV, No. 3, THE AMERICAN MATHEMATICAL MONTHLY, we find

$$c_1=\frac{tpq\sqrt{[2(p^2+q^2)]}}{p+q}, \text{ and } c_2=\frac{t(p^2-q^2)\sqrt{(p^2+q^2)}}{2p}.$$

c_2 will be rational and integral when $p^2+q^2=\square$, and $t=2p$. The general values for p and q, in this case, are $2mn$ and m^2-n^2, $m>n$, $p>q$.

Put $m=2$ and $n=1$. Then $p=4$, $q=3$, and $t=8$.

Whence, $a=96$, $b=28$, $c=100$, and $c_2=35$.

Put $m=3$, and $n=2$. Then $p=12$, $q=5$, and $t=24$.

Whence, $a=1440$, $b=1428$, $c=2028$, and $c_2=1547$.

c_1 will be rational and integral when $p^2+q^2=2\times\square$, and $t=\frac{1}{2}(p+q)$.

The general values for p and q, in this case, are m^2-n^2+2mn and m^2-n^2 the difference between $2mn$, $m>n$ and $p>q$. The numerical results are the same as those obtained in case of c_2, except that a and b have interchanged values.

Of course, only one of the bisectors can be rational in any one triangle.

II. Solution by A. H. BELL, Hillsboro, Ill.

Let the hypotenuse$=h$, and the two legs$=a$ and $2c$, bisector$=b$, and segments $c+s$ and $c-s$.

Then $ah=(c+s)(c-s)+b^2$. $\therefore b^2=ah-(c^2-s^2)$ .... (1).

$a(c+s)=h(c-s)$. $\therefore s=c(h-a)/(h+a)$......(2).

Substituting the general value of the sides of a right triangle in (1) and (2), m^2+n^2, m^2-n^2, $2mn$; $s=n^3/m$, $s^2=n^6/m^2$.

$b^2=(m^2+n^2)(m^2-n^2)^2/m^2=\square$, if m and n are taken for the legs of any integral right triangle. When all the parts are multiplied by m, we obtain integral values, and they are $m(m^2+n^2)$, $m(m^2-n^2)$, and $2m^2n$ for the sides; $\sqrt{[(m^2+n^2)]}(m^2-n^2)$=bisector; $n(m^2\pm n^2)$=segments.

In the right triangle 5, 4, 3, take m and $n=4$ and 3; then the required right triangle is 100, 28, 96; bisector$=35$, segments$=75$ and 21. In the right triangle 13, 12, 5, m and $n=12$ and 5; then the required sides$=2028$, 1428 and 1440, bisector$=1547$, segments$=845$ and 595.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let ABC be a right triangle, right angled at C, AD the bisector of $\angle A$.

Let $BC=a$, $AC=b$, $AB=c$, $AD=d$.

Then $\frac{1}{2}cd\sin\frac{1}{2}A+\frac{1}{2}bd\sin\frac{1}{2}A=\frac{1}{2}bc\sin A$. $cd+bd=2bc\cos\frac{1}{2}A$.

$\therefore d=2bc\cos\frac{1}{2}A/(b+c)$. Also $d=b\sec\frac{1}{2}A$.

$\therefore \sec\frac{1}{2}A=\sqrt{[2c/(b+c)]}$. $\therefore d=b\sqrt{[2c/(b+c)]}$.

Let $a=2pq$, $b=p^2-q^2$, $c=p^2+q^2$. $\therefore \sqrt{[2c/(b+c)]}=\sqrt{(p^2+q^2)}/p$.

Let $p=m^2-n^2$, $q=2mn$. $\therefore \sqrt{[2c/(b+c)]}=\sqrt{(p^2+q^2)}/p=(m^2+n^2)/(m^2-n^2)$.

Let $m=3r$, $n=r$. Then $p=8r^2$, $q=6r^2$. $\therefore a=96r^4$, $b=28r^4$, $c=100r^4$, $d=35r^4$, where r can have any integral value.

IV. Solution by O. S. WESTCOTT, A. M., Sc. D., Principal North Division High School, Chicago, Ill.

$BD : DC :: x^2+y^2 : x^2-y^2$, and $BD+DC$ or $BC : DC :: 2x^2 : x^2-y^2$.

$\therefore\ DC=2xy(x^2-y^2)/2x^2=[(x^2-y^2)/x]y$.

And $AD^2=(x^2-y^2)^2+y^2[(x^2-y^2)^2/x]y^2$.

$AD^2=(x^2+y^2)(x^2-y^2)^2/x^2$. $AD=[(x^2-y^2)/x]\sqrt{(x^2+y^2)}$.

Put $x=4$, $y=3$, and $AD=8\frac{3}{4}$. Increasing each dimension of triangle in quadruple ratio $AD=35$, $AC=28$, $BC=96$, $AB=100$, $DC=21$.

Also solved by *CHAS. C. CROSS*, *JOSIAH H. DRUMMOND*, and *COOPEB D. SCHMITT*.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

128. **Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.**

At what time is the figure 7, on the face of a clock, midway between the hour and minute hands ?

129. **Proposed by J. W. DAPPERT, Civil Engineer and Surveyor, Taylorville, Ill.**

''A Minion, agile, in stature small
Panting came to great Diana's Hall,
Bearing a marble globe upon his shoulders,
Measuring one inch in its diameters.
He rolled it to the northeast corner of the Hall,
Left touching the northern and eastern walls;
Then following came three demi-gods in white,
Each bearing a globe of lustrous metal bright;
One of iron, copper one, and one of silver;
And they placed them in the order given,
Touching each the other, and at the same time,
Touching each the side-walls, in a direct line,
The iron touching the marble, and its other side
Resting against the silver, in its glory and pride,—
All resting upon the oaken floor; and then
With heavy tread, and puff, and roar, Atlas came
Bearing a huge golden sphere, that filled the Hall,
Touching the four sides, floor and ceiling, and all
Radiant with beauty, resting against the silvery ball,
Making the globe's diameters in the rooms diagonal.''

''Tell me, all ye who mathematics know:
What size the copper sphere, and oh!
How large the iron globe ? How great
The golden globe, immaculate ?
The silver sphere, how great ? What size ?
And if presented as a prize,
What value do you hold
Would be the sphere of gold ?''

Found in an old volume entitled "Treatise on Surveying, by Abel Flint"—published in 1833, owned formerly by John E. Stockton, later by S. W. Baker, and now in my possession. Written in red ink upon a blank page.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

ALGEBRA.

117. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Rationalize $l^{\frac{1}{2}}+m^{\frac{1}{2}}+n^{\frac{1}{2}}+x^{\frac{1}{2}}+y^{\frac{1}{2}}+z^{\frac{1}{2}}=0$.

118. Proposed by FREDERIC R. HONEY, Ph. B., Instructor at Trinity College, and Lecturer at Smith College, New Haven, Conn.

An army whose length is equal to a, moves forward. An officer is sent from the rear to the van, and is required to present himself at the rear again when the rear has reached the point where the van was when .the army began to move. How far did the officer travel?

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

GEOMETRY.

141. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

The equilateral triangle described on the hypotenuse of a right triangle is equivalent to the sum of the equilateral triangles described on the other two sides.

Prove without the aid of the famous Pythagorean proposition.

142. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that an infinite number of triangles can be inscribed in $x^2/a^2+y^2/b^2-1=0$ whose sides touch $a^2x^2+b^2y^2=\frac{a^4b^4}{(a^2+b^2)^2}$.

143. Proposed by J. T. FAIRCHILD, A. M., Ph. D., Instructor in Mathematics, Crawfis College, Crawfis College, Ohio.

If the centers of three spheres do not lie in the same straight line, their snrfaces cannot have more than two points in common. These points lie in a straight line perpendicular to the plane of centers and equal distances from this plane on opposite sides. [From *Phillips and Fisher's Geometry.*]

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

CALCULUS.

108. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

The hypotenue of s plane right triangle increases uniformly at the rate of 1-12 of an inch a second. If the legs are as 2 to 3, at what rate is the area of the triangle increasing when the perpendicular from the right angle upon the hypotenuse is 12 inches?

109. Proposed by M. E. GRABER, Heidelberg University, Tiffin, Ohio.

Find the curve in which the product of the perpendicular drawn from two fixed points to any tangent is constant. (To be solyed by using differential equations of the first order.)

110. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

Find the volume removed by boring an auger hole through a right circular cone, the radius of the auger being r, the radius of the cone R, and the altitude h, and the axis of the auger intersecting axis of the cone at right angles and at a distance c from the vertex.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

MECHANICS.

105. Proposed by G. B. M. ZERR. A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $2a$, $2b$ be the diagonals of a rhombus, φ the angle the principal axis at the mid-point of a side makes with the diagonals. Prove $\tan 2\varphi=\frac{3}{5}\tan\beta$, when β is an angle of the rhombus. The principal moments of inertia about this mid-point of the side are $\frac{1}{24}m\{5a^2+5b^2\pm\surd[25(a^2+b^2)^2-64a^2b^2]\}$.

106. Proposed by J. D. CRAIG, A. B., New Germantown, N. J.

The centers of the two wheels of a bicycle are three feet apart.

(1) If a rider wishes the rear wheel to trace a circle 14 feet in diameter, what must be the diameter of the circle traced by the front wheel?

(2) If the rider weighs 120 pounds, and his center of gravity is 4 feet from the ground, at what angle must he lean to make one revolution of the circle every 3 seconds?

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

DIOPHANTINE ANALYSIS.

85. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

Find a square proper fraction which if subtracted from unity will leave for remainder a square proper fraction.

86. Proposed by A. H. BELL, Hillsboro, Ill.

The edges of a rectangular parallelopiped are within one of the proportion $3:6:7$, and if they are $3x$, $6x\mp1$, $7x$, then $(3x)^2+(6x\mp1)^2+(7x)^2=$the diagonal squared$=94x^2\mp12x+1=\square$. To find four integral values of x in this equation.

*** Solutions of these problems should be sent to J. M. Colaw not later than May 10.

AVERAGE AND PROBABILITY.

93. Proposed by LON C. WALKER, Assistant Professor of Mathematics, Leland Stanford, Jr., University, Palo Alto, Cal.

In Problem 75, required the average area of the circle inscribed in the triangle.

94. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Three points are taken at random on the surface of the sphere. Find the chance that the triangle thus formed is acute angled.

*** Solutions of these problems should be sent to B. F. Finkel not later than May 10.

MISCELLANEOUS.

88. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics in University of Tennessee, Knoxville, Tenn.

Sum to infinity the series $5\cos\theta+\dfrac{7\cos3\theta}{3!}+\dfrac{9\cos5\theta}{5!}\ldots\ldots$

89. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Find the earth's average density and mass, having given that the attraction of a ball of lead 2 feet in diameter, on a particle placed close to its surface, is less than the earth's attraction is the ratio 1 : 10250000, and that the density of lead is $11\frac{1}{3}$ times that of water.

90. Proposed by DR. E. D. ROE, Jr., Elmira, N. Y.

I shot my rifle at different ranges and found the following table of elevations e, for the vernier peep sight, for the given distances s:

s	e
0	21.0
100	24.5
200	28.5
300	33.5
400	40.0
500	48.5

The distances are measured in yards. How shall a table of elevations be constructed, giving the arguments e, for every five yards up to 500 yards? Do not give the whole table, but explain the method, and illustrate by giving a computation, carrying the result to three places of decimals. An actual problem.

⁂ Solutions of these problems should be sent to J. M. Colaw not later than May 10.

BOOKS AND PERIODICALS.

Synthetic Arithmetic. By Merritt S. Cook, C. E. 177 pages. Madison, Wis. Tracy, Gibbs & Co. 1899.

The following is the remarkable summary on the title page: "Containing many new principles and improved methods for computation of both simple and compound numbers, multiplication by methods of 'aliquot parts,' complements and partial products, division by substituted divisors, etc.; new method for *squaring* both simple and *compound* numbers, also mixed fractions; the 'basic' system of computing simple interest, by which no direct multiplication by the rate or time is required; also a symmetrical, comprehensive presentation of the Metrical System, aided by the use of algebraic symbols, together with new methods for conversion to and from the English system; finally brief articles on elements and problems connected with electro-motive and water power. Also various miscellaneous problems and new solutions both interesting and useful." We do not think

that the author's "improved methods" of solution will be very favorably received. In his attempt at brevity the writer of the book has "gone off on a tangent" until he is far away from anything that is practical for use in a good text-book. There are some good things in the book, but what is new is for the most part bad, at least for teaching purposes.

J. M. C.

Supplement to Advanced Arithmetic. By A. W. Plummer, M. D., Principal of Olivet Street School, Los Angeles, California. 86 pages. Price, 30 cents. Boston: D. C. Heath & Co. 1898.

This book was prepared to supplement the shortcomings of the Advanced Arithmetic in the California State Series. It contains a practical summary of advanced work, and will no doubt prove very satisfactory to teachers who need to supplement their work in various ways. J. M. C.

Primary Exercises in Arithmetic. Nos. 1, 2, 3 and 4. By H. J. Silver. New York, Cincinnati and Chicago: American Book Company. 1899.

These exercises are intended to supplement those of the text-books. They contain no problems, only the mechanical drill in the fundamental operations. The examples are already set down on the printed page, and the pupil needs only to record the answer in the blank space provided for the purpose. These exercises are well-suited for the little ones.

J. M. C.

Solution Book. By J. T. Fairchild, A. M., Ph. M., Instructor in Mathematics, Crawfis College, Crawfis College, Ohio. 255 pages. Published by the Author. 1898.

In this book solutions are given of a great number of problems that usually give teachers trouble. It covers less ground than some other books of its class, and is more particularly intended to give aid to the common school teacher. The problems have been taken from text-books, mathematical journals, and county examination tests. Several problems and solutions are reproduced from the *Monthly*. Many teachers will want to add this book to their collection. J. M. C.

Advanced Arithmetic. By William W. Speer, District Superintendent of Schools, Chicago. 261 pages. Price 60 cents. Boston: Ginn & Co. 1899.

In the series of which this is the Advanced book experiences in "relating" are at the basis of the treatment. Great prominence is given to the idea of *relative magnitude*, and hence "ratios" are kept constantly in view. The book has many excellent features, but in the attempt to make simple ratios the key to the solution of *all* problems the "ratio" method has been overworked. J. M. C.

Advanced Arithmetic. By E. McN. Carr. 373 pages. Prices, 45 cents. Richmond, Va.: B. F. Johnson Publishing Company. 1899.

Primary Arithmetic. By the same Author. 245 pages. Price, 25 cents.

The treatment of arithmetic as given in these books does not break from the "old methods." The arrangement is topical, and the plan of presentation proceeds upon the theory that principles should be clearly stated, and then illustrated by a reasonable number of appropriate examples and problems. The Primary book would have been better if classification had been made subordinate to gradation, at least in the earlier pages.

J. M. C.

Essentials of Arithmetic. By Gordon A. Southworth, Superintendent of Schools, Somerville, Mass. Book I, for Lower Grades, 186 pages, 40 cents; Book II, for Upper Grades, 311 pages, 60 cents. Boston: Thos. R. Shewell & Company.

These books have been received with remarkable favor. In the selection and arrangement of topics the author has shown a vigorous independence while his methods of presenting the material satisfactorily meets the requirements of the present time. The books are strong in oral work, up to date in business practice, and copious and varied in the supply of practical work and problems. J. M. C.

The Public School Mental Arithmetic. By J. A. McLellan, A. M., LL. D., and A. F. Ames, A. B. 138 pages. Price, 25 cents. New York: The Macmillan Company. 1899.

This book completes the series and completes the *method* of the text-books by the same authors based on the "Psychology of Number." The treatment is in strict line with the idea of number as *measurement*, and perhaps *undue* prominence is given to the *measuring* idea in some of the lessons. The exercises are varied and well graded. J. M. C.

American Elementary Arithmetic. By M. A. Bailey, A. M., Professor of Mathematics in the Kansas State Normal School, Emporia, Kansas. 208 pages. Price 35 cents. New York: American Book Company. 1898.

The pictures in this book are exceptionally good. The grading is not always well done—too much being required of the child in some places. The separate treatment of the fundamental operations is as unhappy as the combined treatment of the Grube method. The book has many valuable suggestions. J. M. C.

Plane Trigonometry and Tables. By Daniel A. Murray, B. A., Ph. D., Instructor in Mathematics in Cornell University, and formerly Scholar and Fellow at Johns Hopkins University. Crown 8vo., 219+95 pages, with a Protractor. Cloth, $1.25. New York, London and Bombay: Longmans, Green and Co. 1899.

The book deals with the subjects considered in the ordinary course in plane trigonometry, in colleges and secondary schools. Careful consideration has been given to the early difficulties and possible future needs of the beginner. In the practical applications, marked attention has been given to the graphical method of solution, as well as to the method of computation. The historical notes throughout the work are a valuable feature. The book is accurately written, and has many points of excellence that justify its appearance. J. M. C.

The following periodicals have been received: *The American Journal of Mathematics*, January, 1900; *The Educational Times*, March 1, 1900; *Journal de Mathématiques Élémentaires*, 1er Mars, 1900; *L'Intermédiaire des Mathématiciens*, Décembre 1899; *The Kansas University Quarterly*, October, 1899; *The Mathematical Gazette*, February, 1900; *Notes and Queries*, February, 1900; *Journal of Education*, January-March, 1900. J. M. C.

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A POPULAR ACCOUNT OF SOME NEW FIELDS OF THOUGHT IN MATHEMATICS.

By DR. G. A. MILLER.

Read at the regular Winter Term Meeting of the Alpha Chapter of Sigma Xi, Cornell University.

At the beginning of the nineteenth century elementary arithmetic was a Freshman subject in our best colleges. In 1802 the standard of admission to Harvard College was raised so as to include a knowledge of arithmetic to the 'Rule of Three.' A boy could enter the oldest college in America prior to 1803 without a knowledge of a multiplication table.* From that time on the entrance requirements in mathematics were rapidly increased, but it was not until after the founding of Johns Hopkins University that the spirit of mathematical investigation took deep root in this country.

The lectures of Sylvester and Cayley at Johns Hopkins University, the founding of the *American Journal of Mathematics* and the young men who received their training abroad coöperated to spread the spirit of mathematical investigation throughout our land. This has led to the formation of the American Mathematical Society eight years ago as well as to the starting of a new research journal, *The Transactions of the American Mathematical Society*, at the beginning of this year. While these were some of the results of mathematical activity, they, in a still stronger sense, tend to augment this activity.

In Europe such men as Descartes, Newton, Leibniz, Lagrange and Euler

*Cajori, *The History and Teaching of Mathematics*, 1890, page 60.

laid the foundation for the development of mathematics in many directions. These men, as well as a few of the most prominent names in the early part of the nineteenth century, were not specialists in mathematics. They were familiar with all the fields of mathematical activity in their day and some of them were well known for their contributions in other fields of knowledge. The last three-quarters of a century and especially the last two or three decades have witnessed a marvelous change in the mathematical activity of Europe. Mathematical periodicals have sprung up on all sides. A number of mathematical societies have been organized and many of the leading mathematicians have confined their investigations to comparatively small fields of mathematics.

The rapid increase of the mathematical literature created an imperative need of bibliographical reviews. This need was met in part by the establishment at Berlin, in 1869, of a year-book devoted exclusively to the reviews of mathematical articles, *Jahrbuch über die Fortschritte der Mathematik.* The 28th volume of this work reached our library a short time ago. It contains over 900 pages, and gives a review of over 2000 memoirs and books. With a view towards further increasing the facilities to keep in touch with the growing literature, the Amsterdam Mathematical Society commenced the publication of a semi-annual review, *Revue Semestrielle*, in 1893. In the last number of this, 236 periodicals are quoted, each of which contains, at times, mathematical articles that are of sufficient merit to be noted. Each of the four countries, France, Germany, Italy, and America publishes over thirty such periodicals.

One of the characterestic features of our times is the prominence of the spirit of coöperation. The mathematical periodicals and the mathematical societies are evidences of this spirit. In quite recent years international mathematical congresses have given further expression of the wide-spread desire to coöperate with even the most remote workers in the same fields. The first of these congresses was held in Zurich in 1897, and the second is to be held during the coming summer in connection with the Paris Exposition. The same spirit led in 1894 to the starting of a periodical, *L'intérmediaire des Mathématiciens*, which is devoted exclusively to the publishing of queries and answers in regard to different mathematical subjects.

This desire for extensive coöperation is tending towards unifying mathematics and towards laying especial stress on those subjects which have the widest application in the different mathematical disciplines. This explains why the theory of functions of a complex variable and the theory of groups are occupying such prominent places in recent mathematical thought.*

Before entering upon a description of some of the fields included in these subjects and the interesting problems which they present, it may be well to state explicitly that our remarks on mathematics will have very little reference to its application to other sciences. To the pure mathematician a result that has extensive application in mathematics is just as important and useful as one which applies to the other sciences. Mathematics is a science which deserves to be de-

*Cf. Klein, *Chicago Mathematical Papers*, 1893, page 134.

veloped for its own sake. The thought that some of its results may find application in other sciences is, however, a continual inspiration, and those who investigate such applications sometimes add materially to the development of mathematics.

The curve representing a function of a single variable was the principal object of investigation during the eighteenth and a great part of the nineteenth century.* The investigations of Abel and Cauchy on power series during the early part of the nineteenth century furnished the foundation for the modern theory of analytic functions—a theory which has been adorned by the labors of some of the most brilliant mathematicians of the preceding generation and which is claiming the attention of some of the foremost thinkers of the present time. Quite recently this theory has been made more accessible to English readers by the treatises of Forsyth and Harkness and Morley.

The critical spirit of our age is, in a large measure, due to the study of the theory of analytic functions. "Newton assumes without hesitation the existence, in every case, of a velocity in a moving point, without troubling himself with the inquiry whether there might not be continuous functions having no derivative."† When it was discovered that such functions exist and that the works of some of the greatest mathematicians of the preceding centuries had to be modified in some instances, mathematicians naturally became much more exacting in regard to rigor, and thus ushered in an age which may be compared with the times of Euclid with respect to its demands for rigor. Whether our critical age will produce a work which, like Euclid, will serve as a model for millenniums cannot be foretold, but it seems certain that works which can stand the critical tests of this age will stand the tests of all ages.

The critical spirit of our times is the foundation of what has been styled the *arithmetization of mathematics*. This movement which the late Weierstrass knew so well to lead is pervading more and more the whole mathematical world. We are rapidly banishing from our treatises the term quantity and replacing it by the word number. Our geometric intuitions are forced into the background and logical deductions from intuitions are taking their places. Who can conceive of curves which have no tangent at any of their rational points in a given interval? Nevertheless it is well known that such curves exist. An account of such functions was first published by Hankel in 1870.‡

Mathematicians find themselves in a great dilemma at this point. Geometric intuition has been such a strong instrument of research and has given so much life and beauty to mathematical investigation that mathematicians cling to it as to their own lives. It is an enormous price when rigor can be purchased only with geometric intuition. Yet, in the present stage of mathematical thought, this seems to be the only thing that will be accepted, and mathematicians stand helpless before this decree.

A few examples may throw some light on this subject. What do we un-

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*Lie, *Leipziger Berichte*, Vol. 47, page 261.
†Klein, *Eveston Colloquium*, 1894, page 41.
‡Cf. Pierpont, *Bulletin of the American Mathematical Society*, Vol. 5, page 398.

derstand by the length of a continuous curve? The intuitionalist says, if we connect different points of the continuous curve by straight lines and find the sum of the lengths of these straight lines, then the length of the curve will be the limit of this sum as the number of the points is indefinitely increased. Jordan was the first to call attention to the fact that this sum need not have a limit. Hence there are continuous curves which do not have any length according to the ordinary definition of length. In fact a number of area-filling curves have recently been studied, and Cantor has shown that a multiplicity of any number of dimensions can be put in a one to one correspondence with a multiplicity of one dimension.

These are some of the facts which have compelled mathematicians to construct their own worlds—the number worlds. Conclusions drawn in one number world do not necessarily apply to another. When a problem is under consideration the number world is so chosen as to meet the demands of the problem. For instance, the constructions and demonstrations of Euclid's geometry seem to require only a space composed of quadratic numbers.* Hence it appears that we do not need to assume that space is continuous in order to demonstrate the theorems of elementary geometry. Similarly in algebra, we are laying more and more stress upon a distinct statement of the number world (the domain of rationality) in which we are operating. Such specifications add a clearness and rigor to our work which would otherwise scarcely be possible.

This refinement which the mathematical thought of to-day is so actively cultivating is not restricted to the finite region. Mathematical infinity is receiving its share of attention. It is well known that it is sometimes desirable to regard the infinite region as a single point. This is, for instance, the case in the transformation known as inversion. Again, in ordinary projective geometry, it is generally convenient to regard the infinite region as of one lower dimension than the finite, so that the infinite region of a plane is merely a line and the infinite region of space is a plane. The student of differential calculus is, moreover, familiar with the infinite variable and the many simplifications which its use makes possible.

The most fruitful investigations along this line are those on multiplicities (Mengenlehre, ensembles). Any total of definite and clearly defined elements is said to form a multiplicity. If two multiplicities are simply isomorphic, *i. e.*, if there is a 1, 1 correspondence between the elements of the multiplicites, they are said to be equivalent, or to have the same power. For example, it is easy to prove that all the positive rational numbers are equivalent to the natural numbers. To do this we may associate all the rational numbers p/q for which the sum $p+q=n$, any positive integer. We thus have the $n-1$ numbers.

$$\frac{n-1}{1},\quad \frac{n-2}{2},\quad \frac{n-3}{3}\cdots,\quad \frac{2}{n-2},\quad \frac{1}{n-1}.$$

We may let 1 correspond to 1; the numbers for which $n=3$ correspond to

*Cf. Strong, *Bulletin of the American Mathematical Society*, Vol. 4, 1898, page 443.

2 and 3 ; the numbers for which $n=4$ correspond to 4, 5 and 6, etc. We thus obtain the following 1, 1 corresponding between all the rational numbers and the positive integers :

$$\tfrac{1}{1}\ ;\ \tfrac{2}{1},\ \tfrac{1}{2}\ ;\ \tfrac{3}{1},\ \tfrac{2}{2},\ \tfrac{1}{3}\ ;\ \tfrac{4}{1},\ \tfrac{3}{2},\ \tfrac{2}{3},\ \tfrac{1}{4}\ ;\ \tfrac{5}{1},\ \tfrac{4}{2},\ \tfrac{3}{3},\ \tfrac{2}{4},\ \tfrac{1}{5}\ ;\ \ldots\ldots$$
$$1\ ;\ 2,\ 3\ ;\ 4,\ 5,\ 6\ ;\ 7,\ 8,\ 9,\ 10\ ;\ 11,\ 12,\ 13,\ 14,\ 15\ ;\ \ldots$$

It may be observed that we do not need to reduce the rational fractions to their lowest terms to effect this correspondence. This method of proof is due to Cantor, who has also proved that all algebraic numbers are equivalent to the natural numbers.* How important and far-reaching the investigations along this line are may be inferred from the fact that Jordan has employed them to serve as a foundation of the elementary part of the second edition of his magistral 'Cours d'analyse.'

A large number of mathematical problems may be reduced to equations involving a single unknown. The solution of such equations has occupied a prominent place in the mathematical literature for centuries. The problem is so difficult that it has been attacked from a number of different points and by means of a large variety of instruments. The instrument which has proved to be the most powerful and far-reaching is substitution groups. By means of it Abel succeeded in 1826 to prove that an equation whose degree exceeds four cannot generally be solved by the successive extraction of roots† and Galois a few years later sketched a far-reaching theory of equations which rests upon the theory of these groups.

In recent years it has been recognized (especially through the labors of Sophus Lie) that the theory of groups has very extensive and fundamental application in a large number of the other domains of mathematics. About a year ago the great French mathematician H. Poincaré, showed in an article, published in the Chicago *Monist*,‡ how this concept may be employed in laying the foundations of elementary geometry. It should be observed that the theory of groups is intrinsically based upon the fundamental concepts of mathematics. It is not a superstructure. It stands on its own foundation and supports more or less a number of other mathematical structures.

As this theory is less known than most of the other extensive branches of mathematics it may be desirable to enter into some details. It is evident that there are rational functions of n independent variables $(x_1, x_2, x_3, \ldots\ldots, x_n)$ which are not changed when these variables are permuted in every possible manner. Such functions are said to be symmetric in regard to these variables. The sum of any given power of these variables $(x_1{}^a+x_2{}^a+x_3{}^a+\ldots\ldots x_n{}^a)$ is an instance of a symmetric function. These functions occupy one extreme. The other extreme is furnished by functions such as $x_1+2x_2+3x_3+\ldots\ldots nx_n$ which change their value for every possible interchange of the variables. Most of the functions are of neither of these extreme types.

*Cantor, *Crelle*, Vol. 77, page 258; cf. Vol. 84, page 250.
†*Crelle*, Vol. 1.
‡October, 1898.

Suppose that a function $\phi(x_1, x_2, \ \ldots\ldots, x_n)$ is not changed by either of two interchanges of its independent variables. Such interchanges are called substitutions and they may be represented by S_1 and S_2. Since ϕ is not changed by either of the substitutions S_1, S_2, it cannot be changed by the substitutions which are equivalent to the succession of these substitutions taken in any order. All such substitutions may be represented by $S_1{}^\alpha\, S_2{}^\beta\, S_1{}^\gamma\, S_2{}^\delta \ldots\ldots$* Since only a finite number of permutations are possible with n letters it follows that $S_1{}^\alpha\, S_2{}^\beta\, S_1{}^\gamma\, S_2{}^\delta \ldots\ldots$ can represent only a finite number of distinct substitutions. The totality of these substitutions is said to be a *substitution group*. Hence we observe that every rational function belongs to some substitution group.

It was soon observed that an infinite number of functions belong to the same substitution group and that all of these functions can be expressed rationally in terms of one of them. The researches of Abel, Galois, and Jordan were based upon these facts and they show that the most important problems in the theory of equations involve the theory of substitution groups. The theory of groups was thus founded with a view to its application to a subject of paramount importance. A broad mathematical subject can, however, not grow vigorously and harmoniously as long as it is studied with a view to its direct applications to other mathematical subjects. It must be free to expand in all directions. That freedom for which the human race has ever been struggling must be vouchsafed to such fundamental subjects before they will exhibit their great fertility and far reaching connections. Less than three years ago the first work on the theory of groups that does not consider its application† was given to the public, but the mathematical journals have been publishing a large number of memoirs along the same line for a number of years.

In defining a substitution group we implied only two conditions; viz. no two substitutions of the group are identical and if we combine the substitutions in any way we obtain only substitutions which are already in the group. Substitutions obey *per se* some other conditions; *i. e.*, when they are combined (multiplied together) they obey the associative law and if we multiply a substitution by (or into) two different substitutions the products will be different. In general we say that any finite number of operations which obey these four conditions constitute a group; *e. g.*, all the rotations around the center of a regular solid which make the solid coincide with itself constitute a group, the n nth roots of unity constitute a group with respect to multiplication but not with respect to addition, etc.

While the bulk of the mathematicians are reveling in the new fields of thought which are opening up on all sides, without any thought in reference to any immediate practical application of their results, there is fortunately a goodly number whose main efforts are devoted towards making some of these new results useful to the investigators in some of the other sciences. As an instance

*The exponent indicates the number of times the substitution is employed in succession.
†Burnside, *Theory of Groups of a Finite Order*, 1897.

of fairly recent work of the latter kind, we may mention the study of linkages with a view towards describing a straight line. Although the straight line is of such fundamental importance both in pure and applied mathematics, yet it seems it was not until the latter half of the nineteenth century that a mechanical device has been discovered by means of which such a line can be described.

In 1864 M. Peaucellier, an officer of engineers in the French army, discovered the well known device to describe a straight line by means of an instrument composed of seven links. "His discovery was not at first estimated at its true value, fell almost into oblivion, and was rediscovered by a Russian named Lipkin, who got a substantial reward from the Russian government for his supposed originality. However, M. Peaucellier's merit was finally recognized and he has been awarded the great mechanical prize of the Institute of France, the Prix Montyon."*

Although the straight line and the circle occupy such a prominent place in elementary geometry and have been before the eyes of the mathematicians for thousands of years, yet less than half a century has passed since the invention of a mechanical device by means of which the straight line can be drawn. Such discoveries go far towards emphasizing the need of investigations even in the most elementary subjects. Such investigations should, however, be preceded by a thorough knowledge of what has been done along the same lines.

If elementary mathematics is to continue to furnish the best possible preparation for the study of advanced mathematics, it is evident that it has to adapt itself to the rapid changes which are going on in the different branches of mathematics. A need is thus created for elementary text-books which meet the new requirements, and we are happy to be able to state that such books are being produced in our midst. How radical such changes may become cannot be foretold. In his address before the New York Mathematical Society, Simon Newcomb said, "The mathematics of the twenty-first century may be very different from our own; perhaps the schoolboy will begin algebra with the theory of substitution groups, as he might now but for inherited habits."† It is to be hoped that our inherited habits will not furnish an insurmountable barrier to progress in this direction.

In modern times the continent of Europe has always been the most progressive and most of the new theories were first developed in these countries. The theory of invariants seems to be an exception to this rule. The two great English mathematicians, Cayley and Sylvester, developed this theory with great vigor; when their important results became generally known on the continent (largely through the work of Clebsch), they aroused a great deal of interest and they furnished a starting point for many important investigations.

One of the fundamental processes of mathematics is transformation—the deducing of truths from given facts and relations. The expressions which remain invariant when given transformations are performed are naturally objects

*A. B. Kempe, *How to Draw a Straight Line*, page 12.

†*Bulletin of the New York Mathematical Society*, 1894, page 95.

of great interest and of fundamental importance. Imbued with this thought Lie once said, "What do the natural phenomena present to us if it is not a succession of infinitesimal transformations of which the laws of the universe are the invariants?"

It need scarcely be added that all mathematical thought even on the same subject is not running in the same channel. Klein has divided mathematicians into three main categories,* viz, the logicians, the formalists, and the intuitionists. The term logician is "intended to indicate that the main strength of the men belonging to this class lies in their logical and critical powers, in their ability to give strict definitions and to derive rigid deductions therefrom. The great and wholesome influence exerted in Germany by Weierstrass in this direction is well known. The formalists among the mathematicians excel mainly in the skillful formal treatment of a given question, in devising for it an algorithm. Gordan, or let us say Cayley or Sylvester, must be ranged in this group. To the intuitionists, finally, belong those who lay particular stress on geometrical intuition, not in pure geometry only, but in all branches of mathematics. What Benjamin Peirce has called 'geometrizing a mathematical question' seems to express the same idea. Lord Kelvin and von Staudt may be mentioned as types of this category."

In his address before the Zurich International Congress Poincaré says,† "Mathematics has a triple end. It should furnish an instrument for the study of nature. Furthermore, it has a philosophic end, and, I venture to say an esthetic end. It ought to incite the philosopher to search into the notions of number, space, and time; and above all, adepts find in mathematics delights analogous to those that painting and music give. They admire the delicate harmony of numbers and of forms; they are amazed when a new discovery discloses for them an unlooked-for perspective; and the joy they experience has it not the esthetic character although the senses take no part in it? Only the privileged few are called to enjoy it fully, it is true, but is it not the same with all the noblest arts? Hence I do not hesitate to say that mathematics deserves to be cultivated for its own sake and that the theories not admitting of application to physics deserve to be studied as well as others. Moreover, a science produced with a view single to its applications is impossible; truths are fruitful only if they are concatenated; if we cleave to those only of which we expect immediate results the connecting links will be lacking, and there will be no longer a chain."

In closing we may remark that no effort has been made to mention all the new fields of mathematical thought. Mathematics, like the other sciences, seems to offer inexhaustable fields of investigation. As it expands its perimeter increases and hence there is a continually increasing demand for investigators. The fields that have been examined present many difficulties which cannot at present be surmounted. Some of the old difficulties are being removed by the light of the new discoveries. Still we know only a few things even about the

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*_The Evanston Colloquium_, page 2.
†_Revue Generale des Sciences_, Vol. 8, page 857.

fields which have been investigated. It is the exception that something can be done by known methods, the rule is that it cannot yet be done.

When we study the literature of some of the older subjects we are sometimes surprised by the large number of known facts, but when we come to study the subjects themselves and ask independent questions we are generally surprised to learn that so few properties are known. Hence it seems very desirable that the advanced student, at least, should study subjects rather than the known facts in regard to these subjects. In this way a more accurate idea of the true state of knowledge can be obtained. Besides the knowledge of having discovered facts and relations which will enter into the structure of a growing science is the greatest source of pleasure that the student can obtain.

Cornell University, March 9, 1900.

AN ELEMENTARY DERIVATION OF THE SERIES FOR sin$x$ and cos$x$.

By H. C. WHITAKER, Ph. D., Philadelphia, Pa.

Assume $\sin x = x + a_3x^3 + a_5x^5 + \ldots\ldots\ldots\ldots$ (1),

$\cos x = 1 + a_2x^2 + a_4x^6 + \ldots\ldots\ldots\ldots$ (2).

Then $\sin y = y + a_3y^3 + a_5y^5 + \ldots\ldots\ldots\ldots$ (3),

$\cos y = 1 + a_2y^2 + a_4y^6 + \ldots\ldots\ldots\ldots$ (4).

Also $\sin(x+y) = x + y + a_3(x+y)^3 + a_5(x+y)^5 + \ldots\ldots\ldots\ldots$ (5),

$\cos(x+y) = 1 + a_2(x+y)^2 + a_4(x+y)^4 + \ldots\ldots\ldots\ldots$ (6).

But $\sin(x+y) = \sin x\cos y + \cos x\sin y \ldots\ldots\ldots\ldots$ (7),

$\cos(x+y) = \cos x\cos y - \sin x\sin y \ldots\ldots\ldots\ldots$ (8).

Hence the coefficients a_2, a_3, a_4 ...... can be determined by equating (5) to $(1)\times(4)+(2)\times(3)$, and by equating (6) to $(2)\times(4)-(1)\times(3)$. In the expansion, in each case, pick out only the terms containing the first power of y. In (5) and (7), the terms are

$$y + 3a_3x^2y + 5a_5x^4y +$$
$$y + a_2x^2y + a_4x^4y +$$

In (6) and (8), the terms are

$$2a_2xy + 4a_4x^3y + 6a_6x^5y +$$
$$-xy - a_3x^3y - a_5x^5y +$$

Equating coefficients of like powers, $2a_2=-1$, $3a_3=a_2$, $4a_4=-a_3$, and $5a_5=a_4$.

Solving for a_2, a_3, a_4, a_5, and substituting in (1) and (2),

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} -$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} -$$

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

[Continued from March Number.]

Writing $s+s^{-1}$ for $\cos 2\varphi$ in (B), (C), (D), (E) we get

$$2(1+e^2-es-es^{-1})^{-\frac{1}{2}}=A_0+A_1(s+s^{-1})+A_2(s^2+s^{-2})+A_3(s^3+s^{-3})$$
$$+\ldots\ldots+A_n(s^n+s^{-n})+\ \ldots\ldots\ldots(37).$$

$$2(1+e^2-es-es^{-1})^{-\frac{3}{2}}=B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+B_3(s^3+s^{-3})$$
$$+\ldots\ldots+B_n(s^n+s^{-n})+\ \ldots\ldots\ldots(38).$$

$$2(1+e^2-es-es^{-1})^{-\frac{5}{2}}=C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+C_3(s^3+s^{-3})$$
$$+\ldots\ldots+C_n(s^n+s^{-n})+\ \ldots\ldots\ldots(39).$$

$$2(1+e^2-es-es^{-1})^{-\frac{7}{2}}=D_0+D_1(s+s^{-1})+D_2(s^2+s^{-2})+D_3(s^3+s^{-3})$$
$$+\ldots\ldots+D_n(s^n+s^{-n})+\ \ldots\ldots\ldots(40).$$

Differentiating (37), (38), and (39) we get

$$e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{3}{2}}=A_1(1-s^{-2})+2A_2(s-s^{-3})+3A_3(s^2-s^{-4})$$
$$+\ldots\ldots+nA_n(s^{n-1}-s^{-n-1})+\ldots.(41).$$

$$3e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{5}{2}}=B_1(1-s^{-2})+2B_2(s-s^{-3})+3B_3(s^2-s^{-4})$$
$$+\ldots\ldots+nB_n(s^{n-1}-s^{-n-1})+\ldots.(42).$$

$$5e(1-s^{-2})(1+e^2-es-es^{-1})^{-\frac{7}{2}}=C_1(1-s^{-2})+2C_2(s-s^{-3})+3C_3(s^2-s^{-4})$$
$$+\ldots\ldots+nC_n(s^{n-1}-s^{-n-1})+\ldots.(43).$$

From (38) and (41) we get

$$\tfrac{1}{2}e(1-s^{-2})[B_0+B_1(s+s^{-1})+B_2(s_2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+\ldots\ldots]$$
$$=A_1(1-s^{-2})+2A_2(s-s^{-3})+3A_3(s^2-s^{-4})+\ldots\ldots+nA_n(s^{n-1}-s^{-n-1})+\ldots\ldots$$

Equating coefficients of s^{n-1} we get

$$2nAn=e(B_{n-1}-B_{n+1})\ldots\ldots\ldots\ldots(44).$$

Writing $(n+1)$ for n we get

$$2(n+1)A_{n+1}=e(B_n-B_{n+2})\ldots\ldots\ldots(45).$$

From (37) and (38) we get

$$(1+e^2-es-es^{-1})[B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+\ldots\ldots$$
$$=A_0+A_1(s+s^{-1})+A_2(s^2+s^{-2})+\ldots\ldots+A_n(s^n+s^{-n})+\ldots\ldots$$

Equating coefficients of s^n we get

$$A_n=(1+e^2)B_n-e(B_{n-1}+B_{n+1})\ldots\ldots\ldots\ldots(46).$$

Writing $(n+1)$ for n we get

$$A_{n+1}=(1+e^2)B_{n-1}-e(B_n+B_{n+2})\ldots\ldots\ldots\ldots(47).$$

Eliminating B_{n-1} between (44) and (46) we get

$$(2n+1)A_n=(1+e^2)B_n-2eB_{n+1}\ldots\ldots\ldots\ldots(48).$$

Eliminating B_{n+2} between (45) and (47) we get

$$(2n+1)A_{n+1}=-(1+e^2)B_{n+1}+2eB_n\ldots\ldots\ldots\ldots(49).$$

Eliminating B_{n+1} between (48) and (49) we get

$$B_n=\frac{2n+1}{(1-e^2)^2}[(1+e^2)A_n-2eA_{n+1}]\ldots\ldots\ldots\ldots(50),$$

$$B_{n+1}=\frac{2n+3}{(1-e^2)^2}[(1+e^2)A_{n+1}-2eA_{n+2}]\ldots\ldots\ldots(51).$$

From (39) and (42) we get

$$\tfrac{3}{2}e(1-s^{-2})[C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+\ldots\ldots+C_n(s^n+s^{-n})+\ldots\ldots$$
$$=B_2(1-s^{-2})+2B_2(s-s^{-3})+\ldots\ldots+nB_n(s^{n-1}-s^{-n-1})+\ldots\ldots$$

Equating coefficients of s^{n-1} we get

$$2nB_n=3e(C_{n-1}-C_{n+1})\ldots\ldots\ldots\ldots(52).$$

Writing $(n+1)$ for n we get

$$2(n+1)B_{n+1}=3e(C_n-C_{n+2}) \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (53).$$

From (38) and (39) we get

$$(1+e^2-es-es^{-1})[C_0+C_1(s+s^{-1})+C_2(s^2+s^{-2})+\ldots\ldots+C_n(s^n+s^{-n})+\ \ldots\ldots$$
$$=B_0+B_1(s+s^{-1})+B_2(s^2+s^{-2})+\ldots\ldots+B_n(s^n+s^{-n})+\ \ldots\ldots$$

Equating coefficients of s^n we get

$$B_n=(1+e^2)C_n-e(C_{n-1}+C_{n+1}) \ldots\ldots\ldots\ \ldots\ldots (54).$$

Writing $(n+1)$ for n we get

$$B_{n+1}=(1+e^2)C_{n+1}-e(C_n+C_{n+2}) \ldots\ldots\ldots\ldots\ldots (55).$$

Eliminating C_{n-1} between (52) and (54) we get

$$(2n+3)B_n=3(1+e^2)C_n-6eC_{n+1} \ \ldots\ \ldots\ldots\ldots (56).$$

Eliminating C_{n+2} between (53) and (55) we get

$$(2n-1)B_{n-1}=-3(1+e^2)C_{n+1}+6eC_n \ \ldots\ldots\ldots\ldots (57).$$

Eliminating C_{n+1} between (56) and (57) we get

$$C_n=\frac{1}{3(1-e^2)^2}[(2n+3)(1+e^2)B_n-2e(2n-1)B_{n+1}] \ \ldots\ldots\ldots (58).$$

Similarly,

$$D_n=\frac{1}{5(1-e^2)^2}[(2n+5)(1+e^2)C_n-2e(2n-3)C_{n+1}] \ \ldots\ldots\ldots (59).$$

And generally,

$$Q_n=\frac{1}{(2m+1)(1-e^2)^2}[(2n+2m+1)(1+e^2)P_n-2e(2n-2m+1)P_{n+1}] \ldots\ldots\ldots (60).$$

(50) and (51) in (58) gives us

$$C_n=\frac{2n+3}{3(1-e^2)^4}[(2n+1)(1+e^2)^2A_n-8en(1+e^2)A_{n+1}+4e^2(2n-1)A_{n+2}] \ \ldots (61).$$

$$C_{n+1}=\frac{2n+5}{3(1-e^2)^4}[(2n+3)(1+e^2)^2A_{n+1}-8e(n+1)(1+e^2)A_{n+2}$$
$$+4e^2(2n+1)A_{n+3}] \ldots\ldots\ldots\ldots\ldots (62).$$

(61) and (62) in (59) will give us

$$D_n=\frac{2n+5}{15(1-e^2)^6}[(2n+3)(2n+1)(1+e^2)^3A_n-6e(1+e^2)^2(2n+3)(2n-1)A_{n+1}$$
$$+12e^2(1+e^2)(4n^2-5)A_{n+2}-8e^3(2n+1)(2n-3)A_{n+3}] \ldots\ldots\ldots (63).$$

Knowing the values of A_n, A_{n+1}, etc., we can find the values of B_n, B_{n+1}, etc.; C_n, C_{n+1}, etc.; D_n, D_{n+1}, etc.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

120. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics in Boys' High School, Reading, Pa.

How many balls 1 inch in diameter can be put in a cubical box 1 foot in the *clear* each way, putting in the maximum number? [From Greenleaf's *Treatise on Algebra.*]

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa., and H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

The maximum number of balls is not 2149, as given Vol. VII, No. 3, but 2151, as demonstrated below.

Put in 4 rows of 12 balls. Then in the space 8×12 can be put 9 more rows of 12 and 11 alternately; for $8\times\frac{1}{2}\sqrt{3}+1=7.928$.

$8-7.928=.072$ of an inch to spare.

This gives in the first layer 9 rows of 12 each$=108$, and 4 rows of 11 each $=44$. $\therefore$ 152 in all.

In the other space $12\times12\times11$ we can put as before eight layers of 144 each and 7 layers of 121 each.

$$\begin{aligned}\therefore\ \text{Eight layers of } 144 &= 1152\\ \text{Seven layers of } 121 &= 847\\ \text{One layer of } 152 &= 152\\ \hline \text{Total} &= 2151\end{aligned}$$

125. Proposed by F. M. PRIEST, Mona House, St. Louis, Mo.

A Quaker once, we understand
For his three sons laid off his land,
And made three equal circles meet
So as to bound an acre neat.
Now in the center of the acre,
Was found the dwelling of the Quaker;
In centers of the circles round,
A dwelling for each son was found.
Now can you tell by skill or art
How many rods they live apart?

I. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The centers of the circles three
With straight lines let united be;
Where touch the arcs, respectively,
These lines will cross the tangency.
Just twice the radius is each line,
And they in trigon space confine
Each circle's sixth and ''acre neat,''
No more nor less. With pencil fleet,
From trigon's several vertices
To circles' opposite tangencies,
Respectively, three uprights trace,

And at their intersections place
The Quakers's dwelling. For we find
These uprights are in each combined
Just two-thirds from the trigon-points
And one-third from the tangent-joints.
Each upright we can plainly see
In radius times square root of three; [$r\sqrt{3}$]
And root of three times radius squared [$r^2\sqrt{3}$]
Is trigon's area unimpaired.
A semicircle interjoined
With the Quaker's acre can be coined
An equal to the trigon's space. [$\frac{1}{2}\pi r^2+160$]
Now equal to each other place
The areas of the trigon found;
And if the work is true and sound,
We'll find the half of sixty-three [31.50+]
Is a trifle less than in rods should be
The radius of each circled bound
Wherein the sons their dwellings found.
Just twice the radius, or sixty-three, [63.0+]
As the rods apart the sons must be.
Two-thirds of the upright, as shown above,
The sons to their father will have to rove;
This distance, in rods, will two decimals run
In one-eighth of two hundred ninety-one. [36.37+]
Now we've told by skill and rhyming art
The number of rods they live apart.

II. **Solution by J. M. HOWIE, State Normal School, Peru, Neb.; LESLIE J. LOCKE, M. A., Fredonia Institute, Fredonia, Pa.; O. S. WESTCOTT, Chicago, Ill.; J. W. DAPPERT, C. E., Taylorville, Ill.; B. L. REMICK, Bradley Institute, Peoria, Ill.; W. MANZILLA, Langston University, Langston, Okla.**

Let ABC be the triangle formed by joining the centers of the farms, CE the altitude. Since the circles are equal, the triangle ACB is equilateral, and therefore $AC=AB=BC=2r$, where r is the radius of the equal circles.

Area of triangle $ABC=\frac{1}{2}AB\times CE=\frac{1}{2}.2r.\sqrt{(4r^2-r^2)}=r^2\sqrt{3}$.

The area of the three circular sectors included in the triangle$=3.\frac{1}{6}\pi r^2=\frac{1}{2}\pi r^2$.

$\therefore$ The area of the curvilinear triangle $EFI=\sqrt{3}r^2-\frac{1}{2}\pi r^2=160$ sq. rods.

$$\therefore r=\sqrt{\left(\frac{320}{2\sqrt{3}-\pi}\right)}=8\sqrt{\left(\frac{5}{2\sqrt{3}-\pi}\right)}=37.7 \text{ rods.}$$

$2r=AC=75.4$ rods=distance between sons' houses, and $\frac{2}{3}\sqrt{3}r=43.5323$ rods=distance from father's to sons' house.

Solved in a similar manner by *G. B. M. ZERR, J. SCHEFFER, C. C. CROSS, H. C. WHITAKER, ELMER SCHUYLER, ALOIS F. KOVARIK, JOHN T. FAIRCHILD, J. M. COLAW, HON. JOSIAH H. DRUMMOND, P. S. BERG, H. I. HOPKINS, COOPER D. SCHMITT*, and *J. O. MAHONEY.*

Solutions of problem 124 were received from *CHAS. C. CROSS, P. S. BERG, J. SCHEFFER, G. B. M. ZERR, ELMER SCHUYLER*, and *H. C. WHITAKER.*

ALGEBRA.

101. **Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.**

Prove that $(1+2+3+....+n)+\frac{n}{2\,!}(1^2+2^2+3^2+....+n^2)+\frac{n(n-1)}{3\,!}$

$(1^3+2^3+3^3+\ldots.+n^3)+\dfrac{n(n-1)(n-2)}{4\,!}(1^4+2^4+3^4+\ldots.+n^4)+\ldots.$

$+\dfrac{n(n-1)(n-2)}{4\,!}(1^{n-3}+2^{n-3}+3^{n-3}+\ldots.+n^{n-3})+\dfrac{n(n-1)}{3\,!}(1^{n-2}+2^{n-2}+3^{n-2}$

$+\ldots.+n^{n-2})+\dfrac{n}{2\,!}(1^{n-1}+2^{n-1}+3^{n-1}+\ldots.+n^{n-1})+(1^n+2^n+3^n+\ldots.+n^n)$

$=(n+1)^n-1$, and substitute for $n=2$, 3, 4, 5, and 6.

Solution by the PROPOSER.

$$(1+x)^{n+1}=1+(n+1)x+\frac{n(n+1)x^2}{2\,!}+\frac{n(n+1)(n-1)x^3}{3\,!}$$

$$+\frac{n(n+1)(n-1)(n-2)x^4}{4\,!}+\ldots.$$

$$\therefore\ \frac{(1+x)^{n+1}-1}{n+1}=x\left(1+\frac{n\,x}{2\,!}+\frac{n(n-1)x^2}{3\,!}+\frac{n(n-1)(n-2)x^3}{4\,!}+\ldots.\right).$$

Let $\dfrac{n}{2\,!}$, $\dfrac{n(n-1)}{3\,!}$, $\dfrac{n(n-1)(n-2)}{4\,!}$, etc., $=a_1$, a_2, a_3, etc., respect.

$$\therefore\ \frac{(1+x)^{n+1}-1}{n+1}=x(1+a_1x+a_2x^2+a_3x^3+\ldots.+a_nx^n).$$

When $x=1$, $\dfrac{2^{n+1}-1}{n+1}=1+a_1+a_2+a_3+\ldots.+a_n$.

When $x=2$, $\dfrac{3^{n+1}-1}{n+1}=2+2^2a_1+2^3a_2+2^4a_3+\ldots.+2^{n+1}a_n$.

When $x=3$, $\dfrac{4^{n+1}-1}{n+1}=3+3^2a_1+3^3a_2+3^4a_3+\ldots.+3^{n+1}a_n$.

$\ldots\ \ldots\ldots\ldots\ldots\ \ldots.\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots$

When $x=n$, $\dfrac{(n+1)^{n+1}-1}{n+1}=n+n^2a_1+n^3a_2+n^4a_3+\ldots.n^{n+1}a_n$

Adding, we get at once,

$$\frac{2^{n+1}+3^{n+1}+4^{n+1}+\ \ldots+(n+1)^{n+1}-n}{(n+1)}$$

$=(1+2+3+4+\ \ldots+n)+(1^2+2^2+3^2+4^2+\ldots.+n^2)a_1+(1^3+2^3+3^3+4^3$

$+\ldots.+n^3)a_2+\ldots.+(1^n+2^n+3^n+4^n+\ldots.4^n+\ldots.+n^n)a_{n-1}$

$+(1n^{n+1}+2^{n+1}+3^{n+1}+4^{n+1}+\ldots\ldots+n^{n+1})a_n.$

Transposing the last quantity in parenthesis, reducing, and replacing the values of a_1, a_2, a_3, we get,

$$(1+2+3+\ldots\ldots+n)+\frac{n}{2\,!}(1^2+2^2+3^2+\ldots\ldots+n^2)+\frac{n(n-1)}{3\,!}(1^3+2^3+3^3+\ldots n^3)$$

$$+\ldots\ldots+\frac{n(n-1)}{3\,!}(1^{n-2}+2^{n-2}+3^{n-2}+\ldots\ldots+n^{n-2})+\frac{n}{2\,!}(1^{n-1}+2^{n-1}+3^{n-1}$$

$$+\ldots\ldots+n^{n-1})+(1^n+2^n+3^n+\ldots\ldots+n^n)=(n+1)^n-1.$$

When $n=2$, we get $(1+2)+(1^2+2^2)=3^2-1=8$.

When $n=3$, $(1+2+3)+\frac{3}{2}(1^2+2^2+3^2)+(1^3+2^3+3^3)=4^3-1=63$.

When $n=4$, $(1+2+3+4)+2(1^2+2^2+3^2+4^2)+2(1^3+2^3+3^3+4^3)$ $+(1^4+2^4+3^4+4^4)=5^4-1=624$.

When $n=5$, $(1+2+3+4+5)+\frac{5}{2}(1^2+2^2+3^2+4^2+5^2)+\frac{10}{3}(1^3+2^3+3^3$ $+4^3+5^3)+\frac{5}{2}(1^4+2^4+3^4+4^4+5^4)+(1^5+2^5+3^5+4^5+5^5)=6^5-1=7775$.

When $n=6$, $(1+2+3+4+5+6)+3(1^2+2^2+3^2+4^2+5^2+6^2)+5(1^3+2^3+3^3$ $+4^3+5^3+6^3)+5(1^4+2^4+3^4+4^4+5^4+6^4)+3(1^5+2^5+3^5+4^5+5^5+6^5)+(1^6+2^6+3^6$ $+4^6+5^6+6^6)=7^6-1=117648$.

Also solved by *ELMER SCHUYLER*.

102. Proposed by J. MARCUS BOORMAN, Woodmere, N. Y.

Solve $2x+\sqrt{[x^2-7]}=5$.

I. Solution by COOPER D. SCHMITT. A. M., University of Tennessee, Knoxville. Tenn.; M. A. GRUBER, A. M., Washington, D. C.; J. SCHEFFER, A. M., Hagerstown, Md ; and G. B. M. ZERR, A. M., Ph. D., Chester High School, Chester, Pa.

Transposing, $\sqrt{(x^2-7)}=5-2x$.

Squaring. $x^2-7=25-20x+4x^2$; whence $3x^2-20x+32=0$.

Solving, $x=4$ or $\frac{8}{3}$.

Neither of these satisfies the original equation, but by writing it thus, $2x-\sqrt{(x^2-7)}=5$, both values will satisfy it.

II. Solution by H. C. WHITAKER. M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia. Pa.

Denote $2x-5$ by y; then the given equation reduces to

$$+\sqrt{(y^2+10y-3)}=-2y.$$

Divide by y, $\quad +\sqrt{(1+10/y-3/y^2)}=-2\ldots\ldots(1).$

But this equation is absurd, since it makes a positive square root equal to a negative number.

Assume the symbolism $+\sqrt{j}=-1$, *j being an impossible quantity, the root of the equation* $+\sqrt{x}+1=0$.

Square (1), $1+10/y-3/y^2=4j$, $y^2(4j-1)-10y+3=0$, from which

$$x=\frac{10j\pm\sqrt{(7-3j)}}{4j-1}.$$

III. Solution by the PROPOSER.

Transpose and square.

$\therefore\ 4x^2-20x+25=x^2-7\ldots\ldots(B)$, an obvious quadratic.

Apply its roots, 4 and $\frac{8}{3}$, to the given (A); hence $2(4)+[-3]=8-3=5$; $\ldots\ldots=2x+\{-[\sqrt{(16-7)}]\}\ldots\ldots(C)$; and

$$2(\tfrac{8}{3})+(-\tfrac{1}{3})=5\tfrac{1}{3}-\tfrac{1}{3}=5\ldots\ldots=2x+\{-[\sqrt{(\tfrac{64}{9}-\tfrac{63}{9})}]\}\ldots\ldots(D);$$

satisfy it. Could extracting $\sqrt{(x^2-7)}$ *positive* here *also* yield roots, then (A)'s dominant *quadratic* (B) is *bi*-quadratic, which is absurd.

Also solved by *P. S. BERG* and *CHAS. C. CROSS.*

GEOMETRY.

127. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

The equation to the plane through the extremities, (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), of conjugate diameters of the ellipsoid,

$$\frac{x^2}{y^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is } \frac{x_1+x_2+x_3}{a^2}x+\frac{y_1+y_2+y_3}{b^2}y+\frac{z_1+z_2+z_3}{c^2}z=1.$$

Solution by the PROPOSER.

If $lx+my+nz=p\ \ldots.(1)$ be the required plane, we should have

$$lx_1+my_1+nz_1=p\ldots\ldots(2),$$
$$lx_2+my_2+nz_2=p\ldots\ldots(3),$$
$$lx_3+my_3+nz_3=p\ldots\ldots(4).$$

Solving these for l/p, m/p, n/p, we have

$$l/p=\begin{vmatrix}1, & y_1, & z_1\\ 1, & y_2, & z_2\\ 1, & y_3, & z_3\end{vmatrix}\div\begin{vmatrix}x_1, & y_1, & z_1\\ x_2, & y_2, & z_2\\ x_3, & y_3, & z_3\end{vmatrix}\ldots\ldots(5).$$

$m/p=$etc., $n/p=$etc., $\ldots\ldots$ (6).

Reducing (5), making use of

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1=\frac{x_2^2}{a^2}+\ldots\ldots=\frac{x_3^2}{a^2}+\quad\ldots\;(7),$$

$$\text{and } x_1y_1+x_2y_2+x_3y_3=y_1z_1+\text{etc.},=z_1x_1+\text{etc.},=0\ldots\ldots(8),$$

$$l/p=\frac{x_1+y_1+z_1}{a^2},\; m/p=\text{etc.},\; n/p=\text{etc.},\;\ldots\ldots(9).$$

These must be put into (1).

Also solved by *G. B. M. ZERR*, *J. W. YOUNG*, *LON C. WALKER*, *J. SCHEFFER*, and *GEORGE LILLEY*.

128. Proposed by W. H. CARTER, Vice President and Professor of Mathematics, Centenary College, Jackson, La.

Given $F=\triangle^{n-1}\div(n-1)!.\triangle_1.\triangle_2\ldots\ldots\triangle_n$, where $\triangle$=the determinant $(a_1b_2c_3\ldots\ldots k_n)$ and $\triangle_1\triangle_2\ldots\ldots\triangle_n$ are the minors of the elements of the nth column; and a_m, b_m, $c_m\ldots\ldots$etc., $(m=1, 2, 3\ldots\ldots n)$ are the coefficients of n given equations containing $n-1$ variables. Show (1) that $n=3$, F=the area of a triangle, and (2) if $n=4$, F=the volume of the tetrahedron.

Solution by J. W. YOUNG, Student in Ohio State University, Columbus, O.

1. Let $n=3$. The points of intersection of the three lines represented by the given equations, are

$$x_1=-\frac{A_1}{C_1};\; x_2=-\frac{A_2}{C_2};\; x_3=-\frac{A_3}{C_3};$$

$$y_1=-\frac{B_1}{C_1};\; y_2=-\frac{B_2}{C_2};\; y_3=-\frac{B_3}{C_3};$$

where, by the usual notation, A_k equals the co-factor a_k, in the determinant $\triangle$.

The area of the triangle formed by these points is

$$\text{Area}=\tfrac{1}{2}\begin{vmatrix} x_1, & y_1, & 1\\ x_2, & y_2, & 1\\ x_3, & y_3, & 1\end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} -\dfrac{A_1}{C_1}, & -\dfrac{B_1}{C_1}, & 1\\ -\dfrac{A_2}{C_2}, & -\dfrac{B_2}{C_2} & 1\\ -\dfrac{A_3}{C_3}, & -\dfrac{B_3}{C_3} & 1\end{vmatrix}=\tfrac{1}{2}\begin{vmatrix} A_1, & B_1, & C_1\\ A_2, & B_2, & C_2\\ A_3, & B_3, & C_3\end{vmatrix}\div C_1C_2C_3$$

and this, by a well-known theorem in determinants,

$$=\tfrac{1}{2}\triangle^2\div C_1C_2C_3=F.$$

2. Let $n=4$. The intersections of the four planes given by the equations are found precisely as above.

The volume of the tetrahedron found by the points is

$$\frac{1}{3\,!}\begin{vmatrix} x_1, & y_1, & z_1, & 1 \\ x_2, & y_2, & z_2, & 1 \\ x_3, & y_3, & z_3, & 1 \\ x_4, & y_4, & z_4, & 1 \end{vmatrix}$$

or substituting the values of $x_1y_1z_1$, etc., we have

$$\text{Volume}=\tfrac{1}{6}\begin{vmatrix} A_1, & B_1, & C_1, & D_1 \\ A_2, & B_2, & C_2, & D_2 \\ A_3, & B_3, & C_3, & D_3 \\ A_4, & B_4, & C_4, & D_4 \end{vmatrix}\div D_1D_2D_3D_4$$

$$=\tfrac{1}{6}\triangle^3\div\triangle_1\triangle_2\triangle_3\triangle_4=F.$$

Also solved by *G. B. M. ZERR*, *WALTER H. DRANE*, and the *PROPOSER*.
Professor Carter asks: What does F represent when n is greater than 4?

CALCULUS.

97. Proposed by ARTEMAS MARTIN, A.M., Ph.D., LL.D., United States Coast and Geodetic Survey Office, Washington, D. C.

An auger hole, radius r, is bored through a prolate spheroid; the axis of the auger passing through the center, perpendicular to the major axis. Find the volume removed.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $\dfrac{x^2}{a^2}+\dfrac{y^2+z^2}{b^2}=1$, be the equation to the prolate spheroid.

$x^2+y^2=r^2$, the equation to the cylinder.

$$\therefore\ V=8/a\int_0^r\int_0^{\sqrt{(r^2-x^2)}}\sqrt{[b^2(a^2-x^2)-a^2y^2]}\,dxdy$$

$$=4/a\int_0^r\sqrt{\{(r^2-x^2)[a^2(b^2-r^2)+(a^2-b^2)x^2]\}}\,dx$$

$$+\frac{4b^2}{a^2}\int_0^r(a^2-x^2)\sin^{-1}\left[\frac{a}{b}\sqrt{\left(\frac{r^2-x^2}{a^2-x^2}\right)}\right]dx$$

$$=4/a\int_0^r\sqrt{\{(r^2-x^2)[a^2(b^2-r^2)+(a^2-b^2)x^2]\}}\,dx$$

$$+\frac{4b^2(a^2-r^2)}{3a}\int_0^r\frac{x^2dx}{\sqrt{\{(r^2-x^2)[a^2(b^2-r^2)+(a^2-b^2)x^2]\}}}$$

$$+\frac{8ab^2(a^2-r^2)}{3}\int_0^r\frac{x^2dx}{(a^2-x^2)\sqrt{\{(r^2-x^2)[a^2(b^2-r^2)+(a^2-b^2)x^2]\}}}$$

Let $x=r\cos\theta$, $\dfrac{r^2(a^2-b^2)}{b^2(a^2-r^2)}=e^2$, $\dfrac{r^2}{a^2-r^2}=c$.

$$V=\frac{4br^2\sqrt{(a^2-r^2)}}{a}\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}\sin^2\theta d\theta$$

$$+\frac{4br^2\sqrt{(a^2-r^2)}}{3a}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$+\frac{8abr^2}{3\sqrt{(a^2-r^2)}}\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}.$$

$$\therefore\ V=\frac{8abr^2(c+1)}{3c\sqrt{(a^2-r^2)}}\Pi(e,c,\tfrac{1}{2}\pi)+\frac{8br^2(a^2-r^2)}{3a}E(e,\tfrac{1}{2}\pi)$$

$$+\frac{4br^2}{3ce^2\sqrt{(a^2-r^2)}}[c(1+e+e^2)(a^2-r^2)-2a^2e^2]\,F(e,\tfrac{1}{2}\pi)$$

$$=\frac{8a^3b}{3\sqrt{(a^2-r^2)}}\Pi(e,c,\tfrac{1}{2}\pi)+\frac{8br^2(a^2-r^2)}{3a}E(e,\tfrac{1}{2}\pi)$$

$$+\frac{4b\sqrt{(a-r^2)}}{3ae^2}[r^2(1+e+e^2)-2a^2e^2]\,F(e,\tfrac{1}{2}\pi).$$

Also solved by *J. SCHEFFER*.

MECHANICS.

87. Proposed by H. C. WHITAKER, M.E., Ph.D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

"He on his impious foes onward drove,
Drove them before him to the bounds
And crystal walls of Heaven; which opening wide
Rolled inward and a spacious gap disclosed
Into the wasteful deep; headlong themselves they threw
Down from the verge of Heaven.
Nine days they fell; Hell at last
Yawning received them whole and on them closed."
Paradise Lost, Book VI.

Assuming Hell to be the center of the earth, and the only force acting on the lost spirits to be that of gravity due to the earth's attraction,—How far is Heaven ?

II. Solution by the PROPOSER.

The published solution of No. 87, Mechanics, assumes (page 82, line 4) that $s=\frac{1}{2}gt^2$ is the formula for falling bodies for great distances.

Neglecting the change in passing from the surface to the center, the formula is $s^3=\dfrac{gr^2t^2}{2\pi^2}$. (See *Watson's Theoretical Astronomy*, page 46, eq. 28).

This formula gives a result somewhat greater than 90 radii of the earth. Assuming 90 radii as correct, the distance the spirits fell in the first second is,

$$x : 16.1 :: 1^2 : 90^2$$

x being .0237 inch.

Assuming that 460,000 radii was correct, the distance fallen in the first second would be .000,000,000,007,2 inch.

NOTE ON PROBLEM 95.

BY FLORIAN CAJORI, PH. D., PROFESSOR OF MATHEMATICS, COLORADO COLLEGE, COLORADO SPRINGS, COLO.

The problem arose in a discussion carried on in the *Nation*, Vol. 68, page 376, between Mr. C. S. Peirce and myself, relating to the validity of an argument given by Galileo and intended to refute the hypothesis that the velocity of a falling body varies as the distance described from a state of rest. Galileo says : "If the velocity with which a body overcomes four yards is double the velocity with which it passed over the first two yards, then the times necessary for these processes must be equal ; but four yards can be overcome in the same time as two yards only if there is an instantaneous motion." Mr. Peirce argues that Galileo's reasoning is sound, a claim which I cannot admit.

The assumption that the velocity shall be proportional to the distance described from the state of rest can be expressed by the formula

$$\frac{ds}{dt}=as, \text{ where } a \text{ is a constant.}$$

Hence the acceleration is $\dfrac{d^2s}{dt^2}=a^2s$. Now initially the distance passed over is zero, *i. e.* $s=0$. Hence the initial accelleration is zero and the body can never begin to move. This conclusion stands even when a is infinitely large, for when absolute zero is a factor, then the product must be zero, no matter how large the other factor may be.* This is the point on which the whole discussion turns. Since Galileo concludes that instantaneous motion is the result, when really there can be no motion at all, his reasoning is fallacious.

But Peirce argues that Galileo used both assumptions stated in the problem, viz., (1) $\dfrac{ds}{dt}=as$, and (2) t finite for a finite distance. Peirce says : ". . . . the solution of the differential equation $\dfrac{ds}{dt}=as$ is $s=Ce^{at}$. In order that s and t should both be zero together, C must be infinitesimal. Then, for a finite value of s, either a or t must be infinite. That is, either the acquired velocity or the time of fall must be infinite. Galileo's argument first adduces the fact that the time is finite, and on that assumption concludes that the hypothesis would in-

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*"In putting together *naughts* to arrive at 1, *we never make any way at all;* the second thousand processes gives no more than the first." DE MORGAN.

volve an infinite acquired velocity, which is absurd.'' In this way Peirce justifies Galileo's conclusion.

The error in Peirce's reasoning seems to me perfectly apparent. When $s=0$ he takes C to be an infinitesimal, while really C is an absolute zero. When s and t are both zero, e^{at} is not zero, hence e^{at}, multiplied by an infinitesimal, cannot be equal to absolute zero. An infinitesimal is a variable whose limit is zero, but the variable never reaches its limit. If we consider an infinitesimal an extremely small quantity, we must still remember that it is a quantity. Now, if C is absolute zero, then s can never be different from zero, no matter how large e^{at} may be.

It is easy to illustrate my conclusion by physical examples. A particle is placed in a smooth tube which revolves horizontally about an axis through its center. With what velocity will the particle move? The only force impelling the particle along the tube is the centrifugal force due to rotation. Hence we have $\frac{d^2s}{dt^2}=w^2s$ and $\frac{ds}{dt}=ws$, where w is the uniform angular velocity. Here the velocity is proportional to the distance from the axis. Suppose now that the particle lies initially at rest *in the axis.* Will it begin to move? There is no reason why it should move one way any more than the other.

The two assumptions in our problem are contraries. The first excludes the possibility of motion; the second declares that motion exists. From assumptions that are contraries no conclusion can be drawn.

DIOPHANTINE ANALYSIS.

80. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Find three square numbers whose reciprocals form an arithmetical progression.

I. Solution by the PROPOSER.

When three numbers are in arithmetical progression, the products of these numbers, taken two at a time, will give three numbers whose *reciprocals* are in arithmetical progression.

Let $x-a$, x, and $x+a$=three numbers in arithmetical progression. Then $x(x-a)$, $(x-a)(x+a)$, and $x(x+a)$=the three numbers whose reciprocals are in arithmetical progression.

$$\text{For, }\frac{1}{x(x-a)}-\frac{1}{(x-a)(x+a)}=\frac{1}{(x-a)(x+a)}-\frac{1}{x(x+a)}=\frac{a}{x(x-a)(x+a)}.$$

The general values for three *squares* in arithmetical progression are $(p^2-q^2-2pq)^2$, $(p^2+q^2)^2$, and $(p^2-q^2+2pq)^2$, where $x=(p^2+q^2)^2$, and a the common difference$=4pq(p^2-q^2)$.

Put $p=2$ and $q=1$; then 1^2, 5^2, and 7^2 are three squares in arithmetical progression. Whence the three squares whose reciprocals form an arithmetical

progression are $5^2=1^2\times5^2$, $7^2=1^2\times7^2$, and $35^2=5^2\times7^2$; and the progression is $\frac{1}{25}$, $\frac{1}{49}$, $\frac{1}{1225}$.

Put $p=3$ and $q=2$; then the required progression is $(\frac{1}{91})^2$, $(\frac{1}{119})^2$, $(\frac{1}{221})^2$.

II. Solution by O. S. WESTCOTT, A. M., Sc. D., Maywood, Ill.

Let x^2, y^2, z^2 represent the numbers. Then $1/z^2-1/y^2=1/y^2-1/x^2$. And $1/z^2+1/x^2=2/y^2$, or $y^2(x^2+z^2)=2x^2z^2$, or $y^2/x^2z^2=2/(x^2+z^2)$.

Since the first member of this equation is a square, the second must be.

$2/(x^2+z^2)=4[2(x^2+z^2)]$, and we have to make $2(x^2+z^2)$ a square.

Put $x=7$ and $z=1$; then $2(x^2+z^2)=2(49+1)=10^2$. Hence the numbers are 49, $\frac{49}{25}$, 1; the progression being $\frac{1}{49}$, $\frac{25}{49}$, 1.

Or put $x=\frac{1}{7}$ and $z=1$; then $2(x^2+z^2)=2[(\frac{1}{49})+1]=(\frac{10}{7})^2$, and the numbers are $\frac{1}{49}$, $\frac{1}{25}$, 1; the progression being 49, 25, 1.

III. Solution by SYLVESTER ROBINS, North Branch, N. J.

Let a^2, x^2 and b^2 represent three squares whose reciprocals $1/a^2$, $1/x^2$, and $1/b^2$ are in arithmetical progression.

Then $1/a^2+1/b^2=2/x^2$, and $x^2=2a^2b^2/(a^2+b^2)$, a square.

Expand $\sqrt{2}=1$, $\frac{7}{5}$, $\frac{41}{29}$, $\frac{239}{169}$, $\frac{1393}{985}$, $\frac{8119}{5741}$, etc.

Say $a=1$; $b=7$, 41, 239, 1393, 8119, etc.

Then 1^2, $(2\times1^2\times7^2)/(1^2+7^2)$, 7^2 .....1, $\frac{49}{25}$, 49.

1^2, $(2\times1^2\times41^2)/(1^2+41^2)$, 41^2......1, $\frac{1681}{841}$, 1681.

1^2, $(2\times1^2\times239^2)/(1^2+239^2)$, 239^2......1, $\frac{57121}{28561}$, 57121.

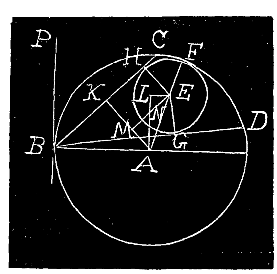
COOPER D. SCHMITT and *G. B. M. ZERR* refer to Problem 78. See solutions of that problem in MONTHLY for March, pages 82—83, by *CHARLES C. CROSS*, *JOSIAH H. DRUMMOND*, *M. A. GRUBER*, and *G. B. M. ZERR*, who also solved the above problem.

AVERAGE AND PROBABILITY.

84. Proposed by L. C. WALKER, Associate Professor of Mathematics, Leland Stanford, Jr.. University, Palo Alto, Cal.

From a point in the circumference of a circle two chords are drawn; find (1) the average radius, and (2) the average area of the circle which touches the two chords and the given circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.



Let $AB=AF=r$, $HE=GE=FE=x$, $\angle PBD=\theta$, $\angle PBC=\varphi$.

Then $BD=2r\sin\theta$, $BC=2r\sin\varphi$, $AK=r\cos\varphi$, $AN=r\cos\theta$, $BH=BG=BK+KH=BN+NG$.

$KH=ME=\sqrt{[(r-x)^2-(r\cos\varphi-x)^2]}$

$=\sqrt{[r^2\sin^2\varphi-2rx(1-\cos\varphi)]}$.

$NG=LE=\sqrt{[(r-x)^2-(r\cos\theta+x)^2]}$

$=\sqrt{[r^2\sin^2\theta-2rx(1+\cos\theta)]}$.

$\therefore\ r\sin\varphi+\sqrt{[r^2\sin^2\varphi-2rx(1-\cos\varphi)]}$

$$=r\sin\theta+\sqrt{[r^2\sin^2\theta-2rx(1+\cos\theta)]}.$$

$$\therefore\ x=\frac{2r\,[(\sin\theta-\sin\varphi)\sin(\theta+\varphi)-(\sin\theta-\sin\varphi)^2]}{(\cos\theta+\cos\varphi)^2}$$

$$=2r[\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)\sec^2\tfrac{1}{2}(\theta-\varphi)-\tan^2\tfrac{1}{2}(\theta-\varphi)].$$

$$\pi x^2=4\pi r^2[\sin\tfrac{1}{2}(\theta+\varphi)\sin\tfrac{1}{2}(\theta-\varphi)\sec^2\tfrac{1}{2}(\theta-\varphi)-\tan^2\tfrac{1}{2}(\theta-\varphi)]^2.$$

Let L=average length, $\triangle$=average area.

$$\therefore\ L=\frac{\int_0^\pi\int_0^\theta x\,d\theta\,d\varphi}{\int_0^\pi\int_0^\theta d\theta\,d\varphi}=\frac{2}{\pi^2}\int_0^\pi\int_0^\theta x\,d\theta\,d\varphi$$

$$=\frac{8r}{\pi^2}\int_0^\pi(\theta\cos^2\tfrac{1}{2}\theta-\sin\theta+\sin\theta\log\sec\tfrac{1}{2}\theta)d\theta=\frac{2r}{\pi^2}(\pi^2-8)=.3789r.$$

$$\triangle=\frac{\pi\int_0^\pi\int_0^\theta x^2\,d\theta\,d\varphi}{\int_0^\pi\int_0^\theta d\theta\,d\varphi}=\frac{2}{\pi}\int_0^\pi\int_0^\theta x^2\,d\theta\,d\varphi$$

$$=\frac{32r^2}{3\pi}\int_0^\pi(6\theta\cos^4\tfrac{1}{2}\theta-3\theta\cos^2\tfrac{1}{2}\theta-6\sin\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta+2\sin^3\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$-12\sin\tfrac{1}{2}\theta\cos^3\tfrac{1}{2}\theta\log\cos\tfrac{1}{2}\theta)d\theta=\frac{4r^2}{3\pi}(3\pi^2-28)=.2174\pi r^2.$$

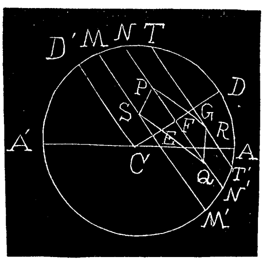
85. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Two points are taken at random in a circle and a chord drawn through them; a point is then taken at random in each segment. Find the average area of the quadrilateral formed by joining the four points.

Solution by the PROPOSER.

Let P, Q, R, S be the four random points taken as indicated in the problem, NN' the chord through PQ, MM', TT' the chords through S, R, respectively.

Draw CD perpendicular and CD' parallel to NN'. Let $CD=r$, $CE=u$, $CF=v$, $CG=w$, $NQ=x$, $PQ=y$, $NF=\sqrt{(r^2-v^2)}=z$, $TG=\sqrt{(r^2-w^2)}=t$, $ME=\sqrt{(r^2-u^2)}=s$, $\angle D'CA'=\theta$.



An element of the circle at P is $dvdx$; at Q, $yd\theta dy$; at R, $2tdw$; at S, $2sdu$.

The limits of θ are 0 and $\frac{1}{2}\pi$ and doubled; of v, $+r$ and $-r$; of x, $2z$ and 0; of y, 0 and x and doubled; of w, v and r and doubled; of u, $-r$ and v and doubled. Area $PRQS=\frac{1}{2}y(w-u)$.

The whole number of ways four points can be taken in the circle is π^4r^8.

$$\therefore \triangle=\frac{16}{\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}\int_v^{r}\int_{-r}^{v}\tfrac{1}{2}y(w-u)d\theta dvdxydy2tdw2sdu$$

$$=\frac{8}{3\pi^4r^8}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}\int_v^{r}[4(a^2-v^2)^{\frac{3}{2}}+6vw\sqrt{(a^2-v^2)}+6a^2w\sin^{-1}(v/a)$$
$$+3\pi a^2w]ty^2d\theta dvdxdydw$$

$$=\frac{16}{3\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}\int_0^{x}(a^2-v^2)^{\frac{3}{2}}y^2d\theta dvdxdy$$

$$=\frac{16}{9\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}\int_0^{2z}(a^2-v^2)^{\frac{3}{2}}x^3d\theta dvdx$$

$$=\frac{64}{9\pi^3r^6}\int_0^{\frac{1}{2}\pi}\int_{-r}^{r}(a^2-v^2)^{\frac{7}{2}}d\theta dv=\frac{35r^2}{18\pi^2}\int_0^{\frac{1}{2}\pi}d\theta=\frac{35r^2}{36\pi}.$$

MISCELLANEOUS.

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74. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

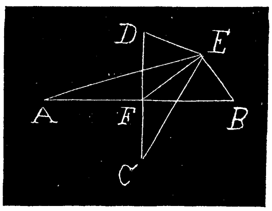
The longest diameter of a horizontal ellipse is $CB=2a=6$ feet. Its shortest diameter is $EF=2b=4$ feet, their intersection being at D. Find in an indefinite vertical plane passing through CB, a point $A=5$ feet$=c$ from D, the ellipse being seen from A as a circle.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

To find a point E in the vertical plane AEB at which DC, AB subtend the same angle, we proceed as follows: Let $DF=FC=b=2$, $AF=FB=a=3$, $FE=c=5$, $AE=x$, $BE=y$, $DE=CE=z$, $\angle DEC=\angle AEB=\theta$.

Then $c=\frac{1}{2}\sqrt{(2x^2+2y^2-4a^2)}=\frac{1}{2}\sqrt{(x^2+y^2+2xy\cos\theta)}$.

$\therefore 2(a^2+c^2)=x^2+y^2$; $4c^2=x^2+y^2+2xy\cos\theta$.



$\therefore x^2+y^2=2a^2+2c^2$ ..... (1).

$2xy\cos\theta=2c^2-2a^2$ ..... (2).

But $c=\frac{1}{2}\sqrt{(4z^2-4b^2)}=\frac{1}{2}\sqrt{(2z^2+2z^2\cos\theta)}$.

$\therefore z=\sqrt{(b^2+c^2)}$ and $\cos\theta=\dfrac{c^2-b^2}{c^2+b^2}$.

$\therefore 2xy=\dfrac{2(c^2-a^2)(c^2+b^2)}{c^2-b^2}$. $\therefore x+y=\dfrac{2\sqrt{(c^4-a^2b^2)}}{\sqrt{(c^2-b^2)}}$, $x-y=\dfrac{2c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}}$.

$$\therefore\ x=\frac{\sqrt{(c^4-a^2b^2)}+c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}},\quad y=\frac{\sqrt{(c^4-a^2b^2)}-c\sqrt{(a^2-b^2)}}{\sqrt{(c^2-b^2)}}.$$

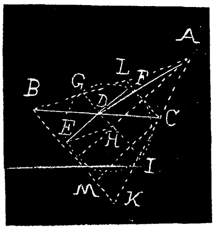
Substituting numbers we get $\cos\theta=\frac{21}{29}$, $\theta=43^\circ\ 36'\ 9''$.

$$x=\frac{\sqrt{(589)}+5\sqrt{(5)}}{\sqrt{(21)}}=7.73575 \text{ feet},\ y=\frac{\sqrt{(589)}-5\sqrt{(5)}}{\sqrt{(21)}}=2.85625 \text{ feet}.$$

$z=\sqrt{(29)}=5.38516$ feet.

IV. **Solution by A. H. BELL, Hillsboro, Ill.**

Let the vertical plane be ABK, A the vertex and BK the base of a right cone. The horizontal cutting plane BC is the major axis of the ellipse with D the projection of the minor axis, the cutting plane GI passing through D, and parallel to the base BK, is a circle and contains the minor axis of the ellipse. Revolving the circle 90° with the diameter GI as an axis, the chord EF is the minor axis of the ellipse; and s, s' are the foci. LC is a circle and parallel to the base BK of the cone.



$BD=DC=a=3$ feet, $ED=DF=b=2$ feet, $AD=c=5$ feet.

The properties of an ellipse give $s,\ s'=BL=CK$......(1).

$BK\times CL=EF^2=4b^2$ ..... (2).

$BC^2=BL^2+BK\times CL$. $\therefore\ BL=2(a^2-b^2)^{\frac{1}{2}}=4.4721360$......(3).

In the right triangle ADF, $AF=AI=AG=(b^2+c^2)^{\frac{1}{2}}=\sqrt{(29)}=5.3851648$.

$BG=GL=CI=\frac{1}{2}BL=2.2360680$.

$AC=AI-CI$. $AB=AI+CI=7.6212328$, $AC=3.1490968$, and the point A is determined.

NOTE. Radius, $GH=(\frac{29}{6})^{\frac{1}{2}}=2.198484326+=\left(\frac{b^2(b^2+c^2)}{2b^2+c^2-a^2}\right)^{\frac{1}{2}}$.

$DH=(GH^2-b^2)^{\frac{1}{2}}=(\frac{5}{6})^{\frac{1}{2}}=0.9128709$.

75. Proposed by J. C. NAGLE, A. M., M. C. E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Texas.

The water tank at the Nacogdoches River on the H. E. & W. T. Ry. is filled by a 3-inch pipe from a reservoir in which the water level is 6 feet above water in tank when full. The top diameter of tank is 17 feet, the bottom diameter is 19 feet, 8 inches, and the pipe projects 10 inches through the bottom. The depth is 13 feet, 6 inches. Find the time required to fill tank, taking the pipe as clean and free from sharp bends, except the right-angled one directly under tank. This bend is 12 feet below outlet of pipe, so that the total length of pipe is 1972 feet. Compare the result with the time of filling if the inlet pipe projected over top of tank instead of entering at the bottom.

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Altitude of cone with top of tank as base$=86\frac{1}{16}$ feet; altitude of cone with bottom of tank as base$=99\frac{9}{16}$ feet; altitude of cone with section 10 inches from bottom as base$=98\frac{35}{48}$ feet.

V=volume of tank$=\pi\{[(59)^3-(51)^3]/(4)^3\}=1136.375\pi$ cubic feet; V_1=volume of portion 10 inches from bottom$=\pi\{[(4779)^3-(4739)^3]/(324)^3\}=79.90614551\pi$ cubic feet. $V_2=V-V_1=1056.46885449$ cubic feet.

Let v=velocity of discharge in feet per second, d=diameter of pipe in feet, l=length, h=head of water, W=weight of water in pounds discharged per second, f=coefficient of friction, β=coefficient of resistance at entrance of pipe, δ=coefficient of contraction at elbow.

Then $h=(v^2/2g)[(fl/d)+\beta+\delta+1]$. (Hydromechanics).

$\therefore\ v=\surd\{2ghd/[fl+d(\beta+\delta+1)]\}$.

$g=32.16$, $l=1972$, $d=\frac{1}{4}$, $f=.030268$, $\beta=.505$, $h=18\frac{2}{3}$, $\delta=.9846$.

$\therefore\ v=.516351\surd h$. $Q=\frac{1}{4}\pi d^2v=.034858\pi$ cubic feet, quantity discharged per second.

$W=62\frac{1}{2}Q=MQ$.

If n is the distance of the top of the tank above the outflow, the resistance or pressure of water in tank on outflow

$$=\tfrac{1}{2}M(\tfrac{1}{64}\pi)\int_o^n dx=\frac{\pi Mn}{128}.$$

$$\therefore\ hW=\frac{v_1^2}{2g}\left[\left(\frac{fl}{d}+\beta+\delta+1\right)W+\frac{\pi Mn}{128}\right].$$

$$\therefore\ 2gh=v_1^2\left(\frac{fl}{d}+\beta+\delta+1+\frac{\pi n}{128Q}\right).$$

$\therefore\ v_1=2.21788$, since $n=12\frac{2}{3}$. $\therefore\ Q_1=\frac{1}{4}\pi d^2v_1=.034654\pi$ cubic feet per second.

T=time$=(V_1/Q)+(V_2/Q_1)=(79.90614551\pi/.034858\pi)+(1056.46885449\pi/.034654\pi)$.

$\therefore\ T=32778.53$ seconds$=9$ hours, 6 minutes, 18.53 seconds.

If filled from the top we will not consider the bend, but suppose pipe 1960 feet long and $h=6$ feet.

Then $h=(v_2^2/2g)[(fl/d)+\beta+1]$.

$\therefore\ v=\surd\{2ghd/[fl+d(\beta+1)]\}=1.2713$.

$\therefore\ Q_2=\frac{1}{4}\pi d^2v_2=.01986\pi$ cubic feet.

t=time$=(V/Q_2)=1136\ 375\pi/.01986\pi=57219.3$ seconds.

$\therefore\ t=15$ hours, 53 minutes, 39.3 seconds. $t-T=6$ hours, 47 minutes, 20.77 seconds.

II. Solution by P. H. PHILBRICK, M.S., C.E., Chief Engineer for Kansas City, Watkins & Gulf Railway Co., Lake Charles. La.

First. To find the time of filling the lower 10 inches of the tank. The head is $h=13.5+6-\frac{5}{6}=18\frac{2}{3}$ feet. Let v=velocity at the outlet, and s=the area of the cross-section of the pipe$=(\frac{1}{4}\pi)(\frac{1}{4})^2=.0491$.

Then $h=v^2/2g+m(v^2/2g)$, or $v=[2gh/(1+m)]^{\frac{1}{2}}$ ......(1), in which m is the sum of the resistances to the entrance and movement of water in the pipe.

The diameter of the tank 10 inches from the bottom is 19.5 feet. Hence the volume of the lower 10 inches of the tank$=(\frac{1}{12}\pi)[(19.67)^2+(19.5)^2+19.67\times19.5]=V$......(2).

Then from (1) and (2), $t=V/vs$..... (3).

According to *Bowser*, Articles 110 and 112, m may be taken$=.03\times1972\div\frac{1}{4}+1.5+.98=238.48$.

Second. To find the time of filling the remainder of the tank. Let x be the height in feet of the top of the reservoir above the surface of the water at the time t. The diameter of the tank at the surface of the water is $17+(\frac{16}{81})(x-6)=15.815+.1975x$. Hence the area is $A=(\frac{1}{4}\pi)(15.815+.1975x)^2=196.4+4.906x+.0306x^2$. Let dx be the rise in the water in the time dt. Then Adx is volume of water admited in time dt ....(4).

We also have, as shown above, $v=[2gx/(1+m)]^{\frac{1}{2}}$, and, therefore, the volume of water admitted in time dt is equal to $[2gh/(1+m)]^{\frac{1}{2}}\times sdt$......(5).

From (4) and (5) we have, $dt=[(1+m)/2g]^{\frac{1}{2}}\times1/s=[(196.4+4.906x+.0306x^2)dx]/x^{\frac{1}{2}}$ .....(6).

Integrating between $x=4=18\frac{2}{3}$ and $x=h=6$, we have $t'=[(1+m)/2g]^{\frac{1}{2}}\times.0491[392.8(H^{\frac{1}{2}}-h^{\frac{1}{2}})+3.271(H^{\frac{3}{2}}-h^{\frac{3}{2}})+.0122(H^{\frac{5}{2}}-h^{\frac{5}{2}})$......(7).

$t+t'$=the total time required to fill the tank.

Third. If the inlet pipe projected over the top of the tank, we would have, as in equation (1), $v_1=[2gh_1/(1+m)]^{\frac{1}{2}}$ ......(8), in which $h_1=6$ feet.

Also volume of tank is $V=(\frac{1}{4}\pi)[(19.67)^2+17^2+17\times19.67]$......(9).

Then $T=V/v_1s$......(10).

76. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that

$$\log[x-a-b\sqrt{(-1)}]=\tfrac{1}{2}\log[(x-a)^2+b^2]-\sqrt{(-1)}\tan^{-1}\frac{b}{x-a},$$

Naperian logarithms being used.

I. Solution by J. O. MAHONEY. B. E., M. Sc., Central High School, Dallas. Tex.; F. ANDEREGG, A. M., Oberlin College, Oberlin, O.; ARTHUR C. LUNN, University of Chicago, Chicago, Ill ; COOPER D. SCHMITT, A.M., University of Tennessee, Knoxville, Tenn.; BURKE SMITH. University of Washington, Seattle, Wash.; H. C. WHITAKER, A. M., Ph. D., Manual Training School, Philadelphia, Pa.

Let $(x-a)-ib=r(\cos\phi-i\sin\phi)$.

Then $r^2=(x-a)^2+b^2$, $\tan\phi=b/(x-a)$.

Thus $\log[x-a-ib]=\log[r(\cos\phi-i\sin\phi)=\log r+\log e^{-i\phi}=\log r-i\phi$

$$=\tfrac{1}{2}\log[(x-a)^2+b^2-i\tan^{-1}[b/(x-a)].$$

II. Solution by HENRY HEATON, M. Sc., Atlantic, Ia.; WALTER H. DRANE, Harvard University, Cambridge, Mass.; and G. B. M. ZERR. A. M., Ph. D., Chester High School, Chester, Pa.

$$\int\frac{\sqrt{(-1)}bdx}{(x-a)^2+b^2}=-\sqrt{(-1)}\tan^{-1}\frac{b}{x-a}=\tfrac{1}{2}\log\left(\frac{x-a-b\sqrt{(-1)}}{x-a+b\sqrt{(-1)}}\right).$$

$$\therefore\ \tfrac{1}{2}\log\left(\frac{x-a-b\sqrt{(-1)}}{x-a+b\sqrt{(-1)}}\right)=-\sqrt{(-1)}\tan^{-1}\frac{b}{x-a}.$$

$$\therefore\ \log[x-a-b\sqrt{(-1)}]-\tfrac{1}{2}\log[(x-a^2)+b^2]=-\sqrt{(-1)}\tan^{-1}[b/(x-a)].$$

$$\therefore\ \log[x-a-b\sqrt{(-1)}]=\tfrac{1}{2}\log[(x-a)^2+b^2]-\sqrt{(-1)}\tan^{-1}[b/(x-a)].$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

130. Proposed by H. C. WHITAKER, M.E., Ph.D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

How many balls 1 inch in diameter can be put in a cubical box 2 feet in the clear each way, putting in the maximum number ?

131. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

A right frustum of a cone whose radii of the bases are r and s, $r>s$, is to be divided into n parts of equal volume by sections parallel to the bases. What are the altitudes of the respective parts ?

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

GEOMETRY.

144. Proposed by L. C. WALKER, Assistant in Mathematics in Leland Stanford, Jr., University, Palo Alto, Cal.

Find the equations of four cones that pass through three given straight lines intersecting in the same point.

145. Proposed by FRANK GRIFFIN, Graduate Student, State University, Boulder, Colo.

If A and B be the points of contact, upon two circles X and Y, of tangents drawn from any point of their circle of similitude, then the tangent from A to Y is equal to the tangent from B to X. [From *Casey's Sequel to Euclid*, Part I., page 114.]

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

AVERAGE AND PROBABILITY.

95. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Three random points are taken in an ellipse, one on each side of the major axis and the third anywhere in the ellipse. Find the average area of the triangle formed by joining the three points.

96. **Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.**

A random straight line is drawn across a square; find the chance that it intersects two opposite sides. [From *Byerly's Integral Calculus*, page 209].

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

MECHANICS.

107. **Proposed by M. E. GRABER, Student, Heidelberg University, Tiffin, O.**

Two particles attracting each other inversely as the square of their distance apart, are constrained to move in straight lines which intersect each other at right angles. How long will it take for the particles to meet and how far does each particle *move ?*

108. **Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechanicsburg, Pa.**

Can it be shown, as a result of Kepler's third law, that the distances are inversely proportional to the squares of the velocities ?

*** Solutions of these problems should be sent to B. F. Finkel not later than June 10.

BOOKS AND PERIODICALS.

The Teaching of Elementary Mathematics. By David Eugene Smith, Principal of the State Normal School at Brockport, New York. 8vo. Cloth, xv+312 pages. Price, $1.00. New York : The Macmillan Company.

In this book may be found the answer to such questions as these: Whence came the subject of mathematics ? Why am I teaching it ? How has it been taught ? What should I read to prepare for my work ? Any book which answers these questions to the entire satisfaction of the teacher and the student is worthy of a high place in the category of educational works. One will find in this work an excellent treatment of the various methods of teaching arithmetic,algebra, and geometry, and in this respect it is of inestimable value to all teachers of elementary mathematics.

Not only is this book of great value to teachers of mathematics, but to all those who have under their direction the formulation of courses of study. It is to be hoped that this work will help to dispel that insane notion yet quite prevalent even among professed educators that mathematics has only utilitarian value, that only so much of it ought to be studied as will be used in the general affiairs of life. Yet these very same educators see no incongruity in spending five or six years in the study of Greek, or Sanskrit, or Hebrew, or Archeology, even though the student expected to study medicine. It is admitted that a large part of practical arithmetic is not generally applicable to ordinary business, and hence is quite impractical, yet it by no means follows that it may not serve a valuable purpose. "Hamlet may bring us neither food nor clothing, and yet a knowledge of Shakespeare is valuable to every one. It is a matter of no moment in the business affiairs of most men that they know where the Caucasus Mountains are, or which way the Rhine flows, or who Cromwell was, and yet we cannot afford to be ignorant of these facts." This is the proper view. Mathematics should be studied and cultivated for its own sake; for the culture and discipline it gives the mind; for the ethical effect its study produces on the mind of man. Since mathematics is becoming more and more the means by which the

phenomena of nature are interpreted, the time is not far distant when one who is ignorant of the principles and processes of the differential and integral calculus will be ignorant of all the advances yet to be made in the fields of natural sciences. Lord Kelvin says that the phenomena of nature are the asymptotes towards which mathematics may approach as near as we please. B. F. F.

Plane and Solid Geometry—Revised Edition. By G. A. Wentworth, Author of a Series of Text-Books in Mathematics. 12mo. Half morocco, 386 pages. Price, $1.25. Boston: Ginn & Company. 1899.

This book has stood the supreme test—successful classroom use, and is recognized as one of the very best works published. In the revised edition the author has retained all the strong points and has added some new ones. The printed page is exceedingly attractive. Full, long-dashed, and short-dashed lines of the figures indicate given, resulting, and auxiliary lines, respectively. Bold-faced, italic, and roman type has been skilfully used to distinguish the hypothesis, the conclusion to be proved, and the proof. The reason for each step is now indicated in small type between that step and the one following, thus avoiding the necessity of interrupting the logical train of thought by turning to a previous section. This help is gradually disgarded as the pupil gains more skill and becomes acquainted with a greater number of geometrical truths. The demonstrations are very clear, and the exercises abundant and interesting. J. M. C.

Algebra for Schools. By George W. Evans, instructor in Mathematics in the English High School, Boston. 427 pages. Price, $1.12. New York: Henry Holt & Company. 1899.

We note in this book with favor the careful classification of problems, and the emphasis given to the several types of equations arising from them; the scheduled explanation of steps in the reduction of equations; the thorough study of literal equations and generalized problems; the application of factoring to the solution of equations; the separation of Elimination into two parts suitable for linear systems and for linear-quadratic pairs respectively; and the great number of graded examples and problems—some 3500 in all—many of which are new. There is some lack of accuracy of statement. "Algebra is a method of abbreviating the explanations of problems in arithmetic. It is also used to abbreviate the statement of rules and the demonstration of theorems." "If equal qualities are multiplied or divided by the same *amount*, they remain equal." He speaks of "nests" of parentheses. We regard the treatment of negative quantities as especially defective and unscientific. In other respects the work is unusually good in arrangement, material, and method of presentation. J. M. C.

Infallible Logic. A visible and automatic system of reasoning. By Thomas D. Hawley, of the Dominion Company, Chicago. 1897.

To the readers of THE AMERICAN MATHEMATICAL MONTHLY the advances made in recent times in Logic are of interest, for they have brought it within the sphere of exact, if not of mathematical analysis. De Morgan started the movement with his *Formal Logic* and his *Memoirs on the Syllogism* and the *Logic of Relatives.* Boole advanced it greatly with his *Laws of Thought*, which he considered his best work; Jevons made a negative advance with his logical machine, and the same may be said of Venn with his *Symbolic Logic.* At the present time Prof. Peano and his colleagues in Italy are applying the Mathematical Logic to express in a concise and exact form all the fundamental definitions and theorems of mathematics. If a young mathematician wishes to become an investigator, he ought by all means to make himself acquainted with the works of the mathematical logicians.

As regards the above work "Infallable Logic," a young mathematician will not get any benefit out of it. It is put forth as a new system, and the author says that "probably

the time is not far distant when it will take the place of the syllogistic and algebraic systems now in current use." It is both amusing and amazing to find that an author who puts forward such claims is ignorant alike of the old logic and of the most elementary mathematics. To prove the former point it is sufficient to mention that he says that the old logic derives by immediate inference from All S is P that Some P is not S; and that he does not see the truth of the fundamental principle of all syllogizing, namely that If all A is B and all B is C then all A is C. To prove the latter point it is sufficient to mention that he gives as an instance of a real proposition "the angles of any triangle are together equal to three angles," and in making an extract from Jevons he confounds differentiation with deduction and integration with induction.

His system is nothing but Jevons' method of writing down all the possible combinations of a number of qualities and their opposites, striking out the combinations which are negatived by the premises and taking what remains as the conclusion. Jevons wrote the combinations in columns on a "logical slate;" Hawley writes them in the form of a multiplication table, which he calls a Reasoning Frame.

Let us see what this invention of a Reasoning Frame can acomplish. Applied (by the author) to the propositions Washington is the capital of the United States.

Salt is chloride of sodium; it yields the following along with two other equally important conclusions:

Either Washington and the capital of the United States or neither is salt and chloride of sodium or neither.

He applies it to elucidate the fallacy "what you bought yesterday you eat today; you bought raw meat yesterday; therefore you eat raw meat today." He puts A=you, B=bought, C=what. D=yesterday E=eat, F=today, G=raw meat, and makes an $ABCD$ EFG table. The conclusion yielded by the Reasoning Frame is rather weak, if indeed, it means anything:

You eat or do not eat today or not today, raw meat. He fails to observe that the combinations he here makes are in no sense whatever the names or characters of one thing.

The Reasoning Frame strikes a snag in the numerically definite syllogism. He quotes the example "If 70 per cent of M are P and 60 per cent are S, then at least 30 per cent are both S and P; and adds "Of course this reasoning is correct, but I am unable to demonstrate its validity by the Reasoning Frame." As a matter of fact the simplest kind of logical diagram suffices. Let the square represent M. Take the most unfavorable case. Suppose that the M which are p are all to the left, and the M which are s are all to the right; there is then 30 per cent of overlapping, hence the M which is both p and s is at least 30 per cent. This principle is written algebraically.

$$sp > s + p - 100$$

Also $sp < s$ and $< p$.

This single principle is worth more than all that is contained in the "Infallible Logic."

Dr. Alexander Macfarlane,
Gowrie Grove, Chatham, Canada.

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NON-EUCLIDEAN GEOMETRY.

By GEORGE BRUCE HALSTED.

In writing of "The Wonderful Century," Alfred Russel Wallace says of all time before the seventeenth century: "Then, going backward, we can find nothing of the first rank except Euclid's wonderful system of geometry, perhaps the most remarkable mental product of the earliest civilizations."

But of late all men of science and intelligent teachers have been hearing more and more of non-Euclidean geometry, and are naturally anxious to know how these new doctrines are related to the traditional geometry which they were taught and perhaps now are teaching.

The new departure is absolutely epoch-making, but fortunately it has intensified admiration for that imperishable model, already in dim antiquity a classic, the immortal *Elements* of Euclid.

But without assumptions nothing can be proved, and Euclid stated his assumptions with the most painstaking candor. He would have smiled at the suggestion that he could ever claim for his conclusions any other truth than perfect deduction from assumed hypotheses.

And so his system is forever safe. Each one of his axioms may turn out to be inconsistent with external reality; each of his fundamental assumptions may be replaced in our final explanation of the space in which we live and move; in reference to our space, all his theorems may be shown to be only approximations; and yet his work will remain a perfect piece of pure mathematics, the exact, eternal geometry of Euclidean space.

For two thousand years no one ever doubted the truth of any one of this

set of axioms, far the most influential in the intellectual history of the world, put together by Euclid in Egypt, but really owing nothing to the Egyptian race, nothing to the boasted lore of Egypt's priests.

The Papyrus of the Rhind, belonging to the British Museum, but given to the world by the erudition of a German Egyptologist, Eisenlohr, and a German historian of mathematics, M. Cantor, gives us more knowledge of the state of mathematics in ancient Egypt than all else previously accessible to the modern world. Its whole testimony confirms with overwhelming force the position that geometry as a science, strict and self-conscious deductive reasoning, was created by the subtle intellect of the same race whose bloom in art still overawes us in the Venus of Milo, the Apollo Belvidere, the Laocoön. But though for twenty centuries the truth of the axioms of the Greek geometer remained unquestioned, there was one of them of which the axiomatic character was doubted even from far antiquity. Elementary geometry was for two thousand years as stationary, as fixed, as peculiarly Greek as the Parthenon. But among Euclid's assumptions is one differing from the others in prolixity, whose place fluctuates in the manuscripts.

Peyrard, on the authority of the Vatican MS., puts it among the postulates, and it is often called the parallel postulate. Heiberg, whose edition of the Greek text is the latest and the best (Leipzig, 1883-1888), gives it as the fifth postulate.

James Williamson, who published the closest translation of the Euclid we have in English, indicating, by the use of italics, the words not in the original, gives this assumption as eleventh among the Common Notions.

Bolyai speaks of it as Euclid's Axiom XI. Todhunter has it as twelfth of the axioms. Clavius (1574) gives it as axiom 13. The Harper Euclid separates it by forty-eight pages from the other axioms.

It is not used in the first twenty-eight propositions of Euclid. Moreover, when at length used, it appears as the inverse of a proposition already demonstrated, the seventeenth, and is only needed to prove the inverse of another propsition already demonstrated, the twenty-seventh.

Geminos of Rhodes (about 70 B. C.) speaks of it as needing proof. The astronomer Ptolemy (A. D. 87-165) tried his hand at proving it. The great Lambert expressly says that Proklus demanded a proof of the assumption because when inverted it is demonstrable. The Arab Nasir-Eddin (1201-1274) tried to demostrate it.

No one had a doubt of the necessary external reality and exact applicability of the assumption. Until the present century the Euclidean geometry was supposed to be the only possible form of space-science; that is, the space analyzed in Euclid's axioms was supposed to be the only non-contradictory sort of space. But could not this assumption be deduced from the other assumptions and the tweth-eight propositions already proved by Euclid without it? Euclid demonstrated things more axiomatic by far. He proves what every dog knows, that any two sides of a triangle are together greater than the third.

Yet after he has finished his demonstration, that straight lines making with the transversal equal alternate angels are parallel, in order to prove the inverse, that parallels cut by a transversal make equal alternate angles, he brings in the unwieldy assumption thus translated by Williamson (Oxford, 1781):

"11. And if a straight line meeting two straight lines make those angles which are inward and upon the same side of it less than two right angles, the two straight lines being produced indefinitely will meet each other on the side where the angles are less than two right angles."

As Staeckel says, "it requires a certain courage to declare such a requirement, alongside the other exceedingly simple assumptions and postulates."

In the brilliant new light given by Bolyai and Lobachevski we now see that Euclid understood the crucial character of the question of parallels.

There are now for us no better proofs of the depth and systematic coherence of Euclid's masterpiece than the very things which, their cause unappreciated, seemed the most noticeable blots on his work.

Sir Henry Savile, in his Praelectiones on Euclid, Oxford, 1621, p. 140, says: "In pulcherrimo Geometriae corpore duo sunt naevi, duae labes . . ." etc., and these two blemishes are the theory of parallels and the doctrine of proportion; the very points in the elements which now arouse our wondering admiration.

But down to our very nineteenth century an ever renewing stream of mathematicians tried to wash away the first of these supposed stains from the most beauteous body of geometry: First, those in which is taken a new definition of parallels. Second, those in which is taken a new axiom different from Euclid's. Third, the largest and most desperate class of attempts, namely those which strive to deduce the theory of parallels from reasonings about the nature of the straight line and plane angle. Hundreds of mathematicians tried at this. All failed. That eminent man Legendre was trying at this, and contiually failing at it, throughout his very long life. Thus the experience of two thousand years went to show that here some assumption was indispensable. Every species of effort was made to avoid or elude it, but without success. From a letter of Gauss we see that in 1799 he was still trying to prove that Euclid's is the only non-contradicting system of geometry, and that it is the system regnant in the external space of our physical experience. The first is false; the second can never be proven.

Yet even in 1831 the acute logician De Morgan accepted and reproduced a wholly fallacious proof of Euclid's assumption, recently republished, Chicago, 1898. A like pseudo-proof published in Crelle's Journal (1834) deceived even our well known Professor W. W. Johnson, who translated and published it in the Analyst (Vol. III, 1876, p. 103), saying, "this demonstration seems to have been generally overlooked by writers of geometrical text-books, though apparently exactly what was needed to put the theory upon a perfectly sound basis."

The most interesting, and perhaps the most extended of such attempted proofs was by the Italian Jesuit Saccheri, born the fifth of September, 1667, who

joined the Society of Jesus at Genoa on the twenty-fourth of March, 1685. He became teacher of grammar in the Jesuit "Collegio di Brera," where the teacher of mathematics was Tommaso Ceva, a brother of the well-known mathematician Giovanni Ceva (1648-1737, who published in 1678 at Milan a work containing the theorem now known by his name.

Saccheri was in close scientific communion with both brothers and received his inspiration from them. He used Ceva's ingenious methods in his first published work, 1693, solutions of six geometric problems proposed by Count Roger Ventimiglia. His attempt at proving the parallel-postulate is his last work.

"Euclid vindicated from every fleck," which received the "Imprimatur" of the inquisition the thirteenth of July, 1733, that of the Provincial of the Jesuits the sixteenth of August, 1733. Saccheri died the twenty-fifth of October, 1733. All preceding attempts were alike in trying to give a direct positive proof of the postulate; all were alike in their assumption open or hidden, conscious or unconscious, of an equivalent postulate.

Saccheri tries a wholly new way, and thus his book marks an epoch. He never doubted the absolute necessary truth of Euclid's postulate, and so he thinks that the two alternatives, possible if it be taken as not true, must each lead to some contradiction, to some absurdity. He tries the reductio ad absurdum. Ninety years later, 1823, Bolyai János reached the astounding conviction that these alternatives lead not to any contradiction but to the "science absolute of space," a generalization of Euclid's universe. In a letter dated the third of November, 1823, written in the Magyar language, and fortunately preserved for us at Maras Vásárhely in Hungary, Bolyai János writes to his father Bolyai Farkas: "I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it. Now I cannot say more, only so much: *That from nothing I have created another wholly new world.*"

Suppose we take a few steps into this new universe on the path which opened before Saccheri without his ever suspecting whither it led.

1. If two points determine a line it is called a straight.

2. If two straights make with a transversal equal alternate angles, they have a common perpendicular.

3. A piece of a straight is called a sect.

4. If two equal coplanar sects are erected perpendicular to a straight, if they do not meet, then the sect joining their extremities makes equal angles with them and is bisected by a perpendicular erected midway between their feet. [Proved by folding the figure over, along the third perpendicular.]

5. Considering figures where the right angles made by the equal perpendiculars may be said to be not alternate, and where no two perpendiculars to the same straight meet, the equal angles made with the joining sect at the extremities of the two equal perpendiculars are either right angles, acute angles, or obtuse angles. Distinguish the three cases as hypothesis of right, hypothesis of acute, hypothesis of obtuse.

6. According to these three hypotheses respectively, the join of the extremities of the equal perpendiculars is equal to, greater than, or less than the join of their feet. [Saccheri, Prop. III. Translated by Halsted in the American Mathematical Monthly.

7. Inversely, according as the join of the extremities is equal to, or less than, or greater than the join of the feet, the equal angles will be right, or obtuse, or acute. [S. P. IV.]

8. Corollary. In every quadrilateral containing three right angles and one obtuse, or acute, the sides adjacent to this oblique angle are less than the opposite sides, if this angle is obtuse, but greater if it is acute.

9. The hypothesis of right, if even in a single case it is true, always in every case it alone is true. [S. P. V.]

10. Assuming the principle of continuity, and referring only to figures where no two perpendiculars to the same straight meet; The hypothesis of obtuse, if even in a single case it is true, always in every case it alone is true. [S. P. VI.]

11. With like limitation; The hypothesis of acute, if even in a single case it is true, always in every case it alone is true. [S. P. VII.]

12. The sum of the angles of the rectilined triangle is a straight angle in the hypothesis of right, is greater than a straight angle in the hypothesis of obtuse, is less than a straight angle in the hypothesis of acute. [S. P. IX.]

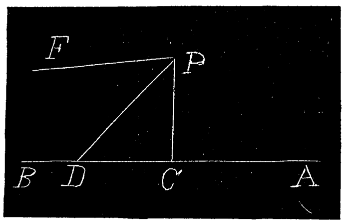
13. The excess of a triangle is the excess of the sum of its angles over a straight angle. The deficiency of a triangle is what its angle-sum lacks of being a straight angle.

14. Two triangles having the same excess or deficiency are equivalent.

15. Even with the assumption that two straights cannot intersect in two points, the three hypotheses give rise to three perfect systems of geometry, the hypothesis of right to Euclid, the hypothesis of acute to Bolyai-Lobachevski, the hypothesis of obtuse to Riemann.

16. In the hypothesis of acute the straight is infinite. Two coplanar straights perpendicular to a third diverge on either side of their common perpendicular. The angle-sum of any rectilineal triangle is less than a straight angle.

17. In Euclid and Bolyai, parallels are straights on a common point at infinity.



18. In Bolyai from any drop point P PC a perpendicular to a given straight AB. If D move off indefinitely on the ray CB the sect PD will approach as limit PF copunctal with AB at infinity. PF is said to be at P the parallel to AB toward B. PF makes with PC an angle CPF which is called the angle of parallelism for the perpendicular PC. It is less than a right angle by an amount which is the limit of the deficiency of the triangle PCD. On the other side of PC an equal angle of parallelism gives us the parallel at P to BA toward A.

Thus at any point there are two parallels to a straight. A straight has two distinct separate points at infinity.

Straights through P which make with PC an angle greater than the angle of parallelism and less than its supplement do not meet the straight AB at all, not even at infinity.

19. A straight maintains its parallelism at all its points. [Lobachevski, Geometrical Researches on the Theory of Parallels, Translated by Halsted, §17.]

20. If one straight is parallel to a second, the second is parallel to the first. [L. §18.]

21. Two straights parallel to a third toward the same part are parallel to each other. [L. §25.]

22. Parallels continually approach each other. [L. §24.]

23. The perpendiculars erected at the middle points of the sides of a triangle are all parallel if two are parallel. [L. §30.]

24. If the foot of a perpendicular slides on a straight, its extremity describes a curve called an equi-distant curve or an equidistantial. An equidistantial will slide on its trace.

25. A circle with infinite radius is not a straight but a curve called the boundary curve, which is a plane curve for which all perpendiculars erected at the mid-points of chords are parallel. [L. §31.] It is an equidistantial whose base line is infinitely removed.

Circles, boundary-curves, equidistantials cut at right angles a system of copunctal straights, of parallel straights, of perpendiculars to a straight, respectively.

Three points determine one of these curves; that is through any three points not costraight will pass either a circle, a boundary-curve, or an equidistantial, and only one such curve.

Any triangle may be inscribed in one and only one of these curves.

26. Boundary-surface we call that surface generated by the revolution of a boundary-curve about one of its axes. Principal plane we call each plane passed through an axis of the boundary-surface.

Every principal plane cuts the boundary-surface in a boundary-curve. Any other plane cuts the boundary surface in a circle.

Boundary-triangles whose sides are arcs of the boundary-curve on the boundary-surface have the same interdependence of angles and sides and the same angle sum as rectilineal triangles in Euclid. Geometry on the boundary surface is the same as the ordinary Euclidean plane geometry. [L. §34.]

27. Triangles on an equidistant-surface are similar to their projections on the base plane; that is, they have the same angles and their sides are proportional.

28. In the hypothesis of obtuse, a straight is of finite size, and returns into itself. This size is the same for all straights. Any two straights can be made to coincide. Two straights always intersect. Two straights perpendicular to a third intersect at a point half a straight from the third either way.

29. A straight in the hypothesis of obtuse does not divide the plane into hemiplanes. Starting from the point of intersection of two straights and passing along one of them over a certain finite sect, we come again to the intersection without having crossed the other straight.

This sect is the whole straight, and so a straight has not really two sides. There is one point through which pass all the coplanar perpendiculars to a given straight. It is called the pole of that straight, and the straight is its polar.

A pole is half a straight from its polar. A polar is the locus of coplanar points half a straight from its pole. Therefore if the pole of one straight lies on another straight, the pole of this second straight is on the first straight.

The cross of two straights is the pole of the join of their poles. The equidistantial is a circle with center at the pole of its basal straight.

Three straights each perpendicular to the other two form a tri-rectangular triangle. It is self-polar, each vertex being the pole of the opposite side.

30. In the hypothesis of obtuse, any two straights enclose a plane figure, a digon. Two digons are congruent if their angles are equal.

31. In the hypothesis of obtuse, all perpendiculars to a plane meet at a point, the pole of the plane. It is the center of a system of spheres of which the plane is a limiting form when the radius becomes equal to half a straight.

Figures on a plane can be projected from similar figures on any sphere which has the pole of the plane for center. They have equal angles and corresponding sides in a constant ratio depending only on the radius of the sphere.

Geometry on a plane is therefore like two-dimensional spherics, but the plane corresponds to only a hemisphere.

The plane is unbounded but not infinite. It is finite in extent. The universe is unbounded but not infinite. It is finite in extent, or content, or volume.

Now of these three possible geometries of uniform space, Euclid's has the unexpected disadvantage that it can never be proved to be the system actual in our external physical world. To establish Euclid, it would be necessary to show that the angle-sum of a triangle is *exactly* a straight angle; and no measurements can ever reach exactitude.

To prove one of the others, we have only to show that the sum of the angles of some triangle is *less than*, or *greater than* a straight angle, which may conceivably be done even by inexact measurements.

What changes ought to be made in teaching elementary geometry in consequence of these later discoveries and the principles of the non-Euclidean geometries?

We are given a new criterion for questions of method, of exposition. For example, surface spherics attains a new importance. When properly founded and expounded, pure spherics, two-dimensional spherics, while giving all the old results and laying the foundation for spherical trigonometry, gives also a picture of the planimetric part of Riemann's geometry, and becomes a touchstone for detecting the fallacies and assumptions in the many pseudo-proofs accepted in the past, such as attempts to found parallelism on direction, attempts to prove all right angles equal, etc.

As another example, we see a new stress laid on the incalculable advantages, educational and scientific, of Euclid's procedure in deducing from three assumed constructions every other construction before he uses it in any demonstration.

The glib method of *supposed* solutions to all desired problems, of hypothetical constructions, is now seen in its deformity and danger. Euclid says, under the heading " Postulates :"

" I. It is assumed, that a straight line may be drawn from any one point to any other point.

" II. And that a terminated straight line [a sect] may be produced in a straight line continually.

"III. And that a circle may be described with any center and radius."

From these Euclid rigidly deduces every problem of construction he wishes to use. Says Helmholtz : "In drawing any subsidiary line for the sake of his demonstration, the well-trained geometer asks always if it is possible to draw such a line. It is notorious that problems of construction play an essential part in the system of geometry.

At first sight these appear to be practical operations, introduced for the training of learners ; but in reality they have the force of existential propositions. They declare that points, straight lines, or circles, such as the problem requires to be constructed, are possible under all conditions, or they determine any exceptions that there may be."

Euclid's first three propositions are problems.

The most popular American geometry, Wentworth's, (1899), puts Euclid's two first postulates on page 8, and the third postulate a whole book later, and then never has a single problem of construction until page 112, where he says : " Hitherto we have supposed the figures constructed."

Meantime, on page 88, he gives as a "theorem : " " Through three points not in a straight line one circumference, and only one, can be drawn."

He gives as his "Proof. Draw the chords AB and BC. At the middle points of AB and BC suppose perpendiculars erected. These perpendiculars will intersect at some point O, since AB and BC are not in the same straight line."

Now the tremendous existential import of the problem, to draw a circle through three non-costraight points, will be recognized when I say that in general it is not possible. In the Lobachevski geometry not every triangle has its vertices concyclic. Granting that every three points must be costraight or concyclic, we could prove the parallel-postulate.

Of the possible geometries we cannot say *a priori* which shall be that of our actual space, the space in which we move.

The hereditary geometry, the Euclidean, is underivable from real experience alone, and can never be proved by experience. Euclidean space is, at least in part, a creation of the human mind. Its adequacy as a subjective form for experience has not yet been disproved.

It can never be proved.

The realities which with the aid of our subjective space form we understand under motion and position, may, with the coming of more accurate experience refuse to fit in that form. Our mathematical reason may decide that they would be fitted better by a non-Euclidean space form.

Comparative geometry finally overthrows that superficial method which pretends to found a logically sound exposition of geometry on "direction," undefined.

For more than twenty years Wentworth gave as his definition "A straight line is a line which has the same direction throughout its whole extent." [1877, Def. 8. 1886, p. 4; 1888, §17.]

At last he discards his aged error, and takes the definition of non-Euclidean geometry, "a straight is the line determined by two points." [1899, §§36 and 46.]

Though the Bolyai and the Riemann geometries are founded on the straight, yet to say in them of two straights that they have the same direction has no ordinary meaning, since in Riemann every two straights cross and inclose a space, while in Bolyai every two parallels continually approach each other. So as to direction, Wentworth has reformed, after twenty years in the land of Nod. But he still says, 1899, §49: "A straight line is the shortest line that can be drawn from one point to another."

Now a relation of equality or inequality between two magnitudes must have some foundation, and be capable of some intelligible test. In the traditional geometry the foundation of all proof by Euclid's method consists in establishing the congruence of magnitudes. To make the congruence evident, the geometrical figures are supposed to be applied to one another, of course without changing their form and dimensions. But since no part of a curve can be congruent to any piece of a straight, so, for example, no part of a circle can be equivalent to any sect from the definition of equivalent magnitudes as those which can be cut into pieces congruent in pairs.

In any comparison of size by congruence, we must be able to place one of the magnitudes or portions of it in complete or partial coincidence with the other. No such direct comparison can be instituted between a straight and a line no piece of which is straight.

Thus the whole of Euclid's *Elements* fails utterly to institute or prove any relation as regards size between a sect and an arc joining the same two points. The operation of measurement we cannot effect, rigorously speaking, either for curves or for curved surfaces, since the unit for length is a sect, and the unit for area, the square on that sect. In fact, however little may be the parts of a curve, they do not cease to be curved, and consequently they cannot be compared directly with a sect; just as parts of a curved surface are not directly comparable with portions of a plane.

We cannot even affirm that any ratio exits between a circle and its diameter until after we have made some extra-Euclidean and post-Euclidean assumption at least equivalent to the following:

No minor arc is less than its chord; and no arc is greater than the sum of

the tangents at its extremities. If the curve be other than a circle we assume that on it one can always take two points so near that the arc between these points is not less than its chord, nor greater than the broken line formed by the two tangents touching its extremities. Some such assumption is, in fact, necessary, but it destroys by itself the primitive idea of measuring curves with straights.

Duhamel gives the assumption the following form: The length of a curve shall be the limit toward which the length of a broken line made up of consecutive chords of that curve approaches, when the number of chords is increased in such a manner that the chords all approach zero as a limit. Thus the elevation of the length of a curve represents not at all an attempt at rectification strictly; but it has for aim the finding of a limit to which another magnitude would approach.

In geometry one proves that as the subdivisions are increased and the sides tend toward the limit zero, the perimeter of the polygon inscribed in a circle increases, circumscribed decreases, toward the same limit, which then is assumed for the magnitude of the circle.

Therefore when Phillips and Fisher, of Yale, give as their definition of a straight [1898, p. 4, §7. Def.] "A *straight line* is a line which is the shortest path between any two of its points," they pass through and beyond Euclid's *Elements* to give us his simplest element; they institute a comparison not only with circular arcs, but also with all curves known and unknown; they presuppose a foreknowledge of all lines in a definition of the simplest line. Is it still needful to say this is grossly bad logic, bad pedagogy, bad mathematics?

The same Yale geometry blunders strikingly on p. 23, where it says: "In fact, Lobatchewsky in 1829, proved that we can never get rid of the parallel axiom without assuming the space in which we live to be very different from *what we know it to be through experience.* Lobatschewsky tried to imagine a different sort of universe in which the parallel axiom would not be true. This imaginary kind of space is called *non-Euclidean* space, *whereas the space in which we really live* is called *Euclidean*, because Euclid (about 300 B. C.) first wrote a systematic geometry of *our space.*"

The scientific doctrine of evolution postulates a world independent of man, and teaches the outcome of man from lower forms of life in accordance with wholly natural causes. In this world of evolution experience is a teacher, but man is a creator, and the mighty examiner is death.

The puppy born blind must still be able, guided by the sense of smell, to superimpose his mouth upon a source of nourishment. The little chick, responding to the stimulus of a small bright object, must be able to bring his beak into contact with the object so as to grasp and then swallow it. The springing goat that misjudges an abyss is lost.

So too with man. His ideas must in some way correspond to this independent world, or death passes upon him an adverse judgment. But it is of the very essence of the doctrince of evolution that man's metric knowledge of this

independent world, having come by gradual betterment and through imperfect instruments, for example the eye, cannot be absolute and exact.

The results of any observations are always with certain definite limitations as to exactitude and under particular conditions. Man the creator replaces these results by assumptions presumed to have absolute precision and generality, such as, for example, the so-called axioms of Euclid.

If two natural hard objects, susceptible of high polish, be ground together, their surfaces in contact may be so smoothed as to fit closely together and slide one on the other without separating. If now a third surface be ground alternately against each of these two smooth surfaces until it accurately fits both, then we say that each of the three surfaces is approximately plane, is a piece of a plane. If one such plane be made to cut through another, we say the common line where they cross is approximately a straight. The perfect, the ideal plane, is a human creation under which we seize the imperfect data of experience.

If three approximate planes on real objects be made to cut through a fourth approximate plane, then three approximate straights are formed on this fourth plane, and in general they are found to intersect, and the figure they make we may call an approximate triangle. Such triangles vary greatly in shape. But no matter what the shape, if we cut oft the six ends of any two such, and place them side by side on a plane with their verticles at the same point, the six are found with a high degree or approximation just to fill up the plane about the point. If the whole angular magnitude about any point in a plane be called a perigon, then we may say that the six angles of any two approximate triangles are found to be together approximately a perigon.

Now does the exactness of this approximation to a perigon depend only on the straightness of the sides of the original two triangulars, or also upon their size?

If we know with absolute certitude that the size of the triangles has nothing to do with it, then we know something that we have no right to know according to the doctrine of evolution, something impossible for us ever to have learned evolutionally.

Yet before the epoch-making ideas of Bolyai János and Lobachevski, every one supposed we were perfectly sure that the angle-sum of an actual approximate triangle approached a straight angle with an exactness dependent only on the straightness of the sides and not at all on the size of the triangle. But if in the mechanics of the world independent of man we were absolutely certain that all therein is Euclidean and only Euclidean, then Darwinism would be disproved by the reductio ad absurdum.

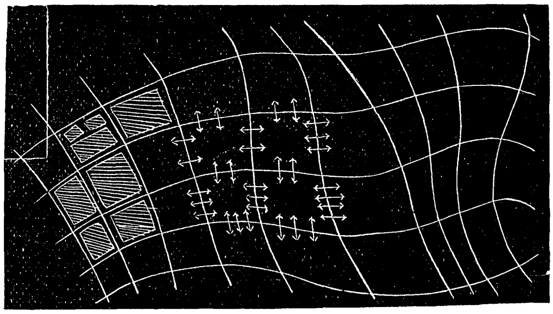
All our measurements are finite and approximate only. The mechanics of actual bodies in what Cayley called the external space of our experience, might conceivably be shown by merely approximate measurements to be non-Euclidean, just as a body might be shown to weigh more than two grams or less than two grams, though it never could be shown to weigh precisely, absolutely two grams.

The outcome of the non-Euclidean geometry is a new freedom to explain and understand our universe and ourselves.

ON THE PROJECTIVITY OF STRESSES IN A PLANE.

By ARNOLD EMCH, Ph. D., The University of Colorado.

1. For geometrical purposes we may assume that a solid having the shape of a slab and subjected to stresses parallel to its forces reduces to a plane surface of infinitesimal thickness. If we now consider any point A of the plane, and the pencil of rays passing through this point, every ray represents a section which is subject to definite strains. Let s be such a ray and $\triangle s$ a small line-element of s at the point A, and $\triangle R$ the resultant of the strains acting upon $\triangle s$; then $\frac{\triangle s}{\triangle R}$ is called the *specific stress* in the line element $\triangle s$ and is evidently also expressed by



$$\frac{ds}{dR}=\lim\left(\frac{\triangle s}{\triangle R}\right)=\rho, \text{ say.}$$

In general the quantity ρ is different for every section and may assume a maximum and minimum at a point A. The laws of specific stresses of all sections through A may easily be obtained by the methods of graphic statics* and are as follows:

(a). *If ρ is the specific stress acting at A upon a section S, then the specific stress ρ' acting upon a section S' which is parallel to ρ, is parallel to S; i. e., if S' is parallel to the direction of ρ, then ρ' is parallel to the direction of S.*

(b). *All pairs of rays S and S' defined by (a) form an involution of rays. As each involution contains a rectangular pair, r perpendicular to r', it follows that at each point A there are two perpendicular sections each of which is perpendicular to its corresponding specific stress.*

(c). *If the involution has double rays, it is evident that there are only shearing stresses along these sections. These rays separate the sections in which the material is only subjected to tensions from those subjected to compression only. If one ray S is subjected to tension, then the corresponding ray S' is subjected to compression only. The same is true of the rectangular pair (r, r').*

(d). *If the double rays of the involution are imaginary, then all sections are snbjected to only one kind of stresses, either tension or compression. In this case there are two perpendicular sections at A which are subjected to normal stresses of the same kind.*

2. In both kinds of involution it is now possible to construct two systems of curves in a plane, so that all tangents of these curves, representing sections through the material, are perpendicular to their corresponding stresses.

*See Ritter, *Anwenendungen Graphischen Statik*, Vol. I, pages 1—25, Zurich.

In the case of an hyperbolic involution (c) these curves consist of two orthogonal systems of curves of normal compressions and tensions. If the involution is elliptic the orthogonal systems consist of curves whose tangents are all subjected to the same kind of normal stresses, *i. e.*, either to tension or to compression only. The first case is well known in graphic statics. Little attention has, however, been paid to the elliptic involution of stresses in a plane, and it will be interesting to illustrate this case by a few examples which strikingly exhibit the character of orthogonality of the curves of normal tensions.

If the material of a slab is only subjected to external tensions, the curves of normal tensions will resemble those of the adjoining figure.

As soon as the stresses in certain portions of the material exceed the strength of the material, the rupture of the material will take place along the curves of normal tensions, as it is evident from the figure.

This fact is beautifully illustrated by the cracks that form in a drying mass of mud along a river, where by the contraction of the mass only tensions are produced. It may also be observed on a heavily varnished surface, and in numerous other examples which depend only on tension.

In a future article the author intends to find orthogonal systems of normal stresses for a number of cases where the law of the distribution of specific stresses in a plane is a given function of the coördinates and of the angle of inclination of the line-element at a point.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

126. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Bought 150 head of stock for $300, paying for each kind $2 5-6, $1 5-9, and $5-7, respectively. Find number of each kind bought.

I. Solution by MARCUS BAKER, U. S. Coast and Geodetic Survey, Washington, D. C.; H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; and CHARLES C. CROSS, Meredithville, Va.

The conditions are $x+y+z=150$, $2\frac{5}{6}x+1\frac{5}{9}y+\frac{5}{7}z=300$, and these to be solved in positive integers.

Eliminating x we have $\frac{23}{18}y+\frac{89}{42}z=125$; or

$$y=97-z+\frac{133-106z}{161}.\quad \text{Put } 133-106z=161a.$$

$\therefore\ z=1-a+\dfrac{27-55a}{106}$. Put $27-55a=106b$.

$\therefore\ a=-b+\dfrac{3(9-17b)}{55}$. Put $9-17b=55c$.

$\therefore\ b=-3c+\dfrac{9-4c}{17}$. Put $9-4c=17d$.

$\therefore\ c=2-4d+\dfrac{1-d}{4}$. Put $1-d=4n$.

$\therefore\ d=1-4n$; and by substituting through the various steps,
$c=17n-2$,
$b=7-55n$,
$a=106n-13$,

$$\left.\begin{array}{l} z=21-161n \\ y=63+267n \\ x=66-106n \end{array}\right\}$$ Here n may have *any* value, but $n=0$ alone gives positive integral values for x, y and z.

Therefore he bought of the

First kind,	66 at $\$2\frac{5}{6}$,	\$187
Second kind,	63 at $\$1\frac{5}{9}$,	98
Third kind,	21 at $\$\ \frac{5}{7}$,	15
	150	\$300

II. Solution by S. F. NORRIS, Professor of Astronomy and Mathematics, Baltimore, Md.; SYLVESTER ROBBINS, North Branch, N. J.; P. S. BERG, B. S., Principal of Schools, Larimore, N. D.; M. A. GRUBER, A. M., War Department, Washington, D. C.; and ARCHIE C. MURRY, Baltimore, Md.

Average price $=\$2$.

	1st	2nd			Res't
$2\frac{5}{6}$	8	54	$33\frac{3}{5}$	$32\frac{2}{5}$	66
$1\frac{5}{9}$	15		63		63
$\frac{5}{9}$		35		21	21

Balancing gains and losses :

$$\left.\begin{array}{l}\text{On 1 animal at } \$2\tfrac{5}{6},\text{ lose } \$\tfrac{5}{6}\text{; hence buy 8 animals at } \$2\tfrac{5}{6}. \\ \text{On 1 animal at } \$1\tfrac{5}{9},\text{ gain } \$\tfrac{4}{9}\text{; hence buy 15 animals at } \$1\tfrac{5}{9}.\end{array}\right]\text{1st.}$$

$$\left.\begin{array}{l}\text{On 1 animal at } \$2\tfrac{2}{6}.\text{ lose } \$\tfrac{5}{6}\text{; hence buy 54 animals at } \$2\tfrac{5}{6}. \\ \text{On 1 animal at } \$\tfrac{5}{7},\text{ gain } \$\tfrac{9}{7}\text{; hence buy 35 animals at } \$\tfrac{5}{7}.\end{array}\right\}\text{2nd.}$$

NOTE.—To get required number, multiply 1st column by $4\frac{1}{5}$, and 2nd column by $\frac{3}{5}$.

Answer.	66 animals at $\$2\frac{5}{6}=$	\$187
	63 animals at $1\frac{5}{9}=$	98
	21 animals at $\frac{5}{7}=$	15
	150 animals for	\$300

ALGEBRA.

103. Proposed by **WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.**

Given the equation $x^m+p_1x^{m-2}+p_2x^{m-2}+\dots p_{m-1}x+p_m=0$ freed from multiple roots. Prove that its discriminant is positive or negative according as the number of pairs of complex roots is even or odd.

I. Solution by the **PROPOSER.**

We have
$$\begin{aligned}\triangle=(x_1-x_2)^2(x_1-x_3)^2(x_1-x_4)^2&\dots(x_1-x_m)^2\\ (x_2-x_3)^2(x_2-x_4)^2&\dots(x_2-x_m)^2\\ (x_3-x_4)^2&\dots(x_3-x_m)^2\\ &\dots\dots\dots\\ &(x_{m-1}-x_m)^2.\end{aligned}$$

There are seven cases to be considered :

I. Differences of real roots.
II. Differences of a real and a complex root.
III. Differences of a real and a pure imaginary root.
IV. Differences of two pure imaginaries.
V. Differences of a pure imaginary and a complex.
VI. Differences of two complex roots not conjugate.
VII. Differences of two conjugate complex roots.

I. It is evident at once, since each difference is squared, that the real roots alone can not affect the sign.

II. If c be a real root, and $a+bi$ a complex, for every difference of the form $c-(a+bi)$ we shall also have one of the form $c-(a-bi)$ and as the product of these two is positive, their square is also positive, and hence case II can not affect the sign of $\triangle$.

By exactly the same reasoning it can be shown that cases III, V, and VI cannot affect the sign of $\triangle$.

IV is a special form of VII, so we shall pass to the last. Let $a+bi$ and $a-bi$ be two conjugate complex roots. Then the difference is $\pm bi$ whose square is $-b$. Hence for every pair of roots we use in forming a difference, we change the sign of $\triangle$, and hence $\triangle$ is positive or negative according as the number of pairs of complex roots is even or odd.

II. Solution by **J. W. YOUNG, Fellow and Assistant in Mathemetics, Ohio State University, Columbus, O.**

The discriminant is equal to the product of squares of the differences of roots taken in all possible combinations, *i. e.*: $\triangle=\Pi(x_1-x_2)^2$ $[x, x_2, x_3\dots x_n$ $=$roots of equation].

For every complex pair of roots, there will be a factor of the form

$$[(\xi+i\eta)-(\xi-i\eta)]^2=[2i\eta]^2=-4\eta^2.$$

That is, for every complex pair there will be a *negative* factor in the discriminant.

Moreover the remaining factors will give you a *positive* product.
For, if one of the factors be

$$[(\xi_1+i\eta_1)-(\xi_2+i\eta)]^2=[(\xi_1-\xi_2)+i(\eta_1-\eta_2)]^2$$

there will be a corresponding factor

$$[(\xi_1-i\eta_1)-(\xi_2-i\eta_2)]^2=[(\xi_1-\xi_2)-\mathrm{i}(\eta_1-\eta_2)]^2.$$

When multiplied these give

$$[(\xi_1-\xi_2)^2+(\eta_1-\eta_2)^2]^2,$$

which is clearly positive. So all the factors containing complex roots may be taken in pairs which give a positive product, and, of course, the factors containing real roots only are positive. Hence corresponding to every pair of complex roots there will be *one* and *only one* negative factor. Therefore, if the number of pairs be even the discriminant will be positive ; if odd, negative.

NOTE.=Mr. Harry S. Vandiver should have been credited as a joint author of the solution of problem 102.

GEOMETRY.

129. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

Show that at no point of an ellipse will the circle of curvature pass through the center, if the eccentricity be less than $\frac{1}{2}\sqrt{2}$.

I. Solution by F. ANDEREGG, A.M., Professor of Mathematics, Oberlin College, Oberlin, O.; W.H. CARTER, A. M., Professor of Mathematics, Centenary College, Jackson, La.

Since for the ellipse the radius of curvature is

$$\rho=\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4};$$

and the center of curvature is the point

$$\left(\frac{(a^2-b^2)x^3}{a^4},\ -\frac{(a^2-b^2)y^3}{b^4}\right),$$

the equation of the circle of curvature is

$$\left(x'-\frac{(a^2-b^2)x^3}{a^4}\right)^2+\left(y'+\frac{(a^2-b^2)y^3}{b^4}\right)^2=\frac{(a^4y^2+b^4x^2)^3}{a^8b^8}.$$

This circle passes through the origin

$$(a^2-b^2)^2(b^8x^6+a^8y^6)=(a^4y^2+b^4x^2)^3.$$

This equation is easily simplified, and assumes the form

$$\frac{x^2}{a^2}=\frac{1+e^2}{3e^2}.$$

It follows at once that the least value of $3e^2$ is $1+e^2$, or the least value of e is $\frac{1}{2}\sqrt{2}$.

II. Solution by the PROPOSER.

If ϕ be the eccentric angle of any point of the ellipse $a^2y^2+b^2x^2=a^2b^2$, the equation to the corresponding circle of curvature is

$$x^2+y^2-(a^2-b^2)\left(\frac{2\cos^3\phi}{a}x-\frac{2\sin^3\varphi}{b}y\right)$$

$$+a^2(\cos^2\varphi-2\sin^2\varphi)-b^2(2\cos^2\varphi-\sin^2\varphi)=0.$$

This passing through the center, requires that

$$a^2(\cos^2\phi-2\sin^2\varphi)-b^2(2\cos^2\varphi-\sin^2\varphi)=0,$$

$$\text{or } \frac{a^2-b^2}{a^2}=e^2=\frac{1}{2-3\sin^2\varphi},$$

which is a minimum for $\phi=0$ or π; that is $e^2=\frac{1}{2}$.

Excellent solutions were received from *J. W. YOUNG*, *G. B. M. ZERR*, *J. SCHEFFER*, and *W. H. DRANE*.

CALCULUS.

98. Proposed by CHARLES CARROLL CROSS, Meredithville, Va.

Of the circumference of a fixed circle radius R rolls a circle radius r. Required the length of the curve described by a point on the circumference of the rolling circle; (1) when the circle rolls on the inside; (2) when the circle rolls on the outside of the circumference of the fixed circle.

Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee. Knoxville, Tenn.; WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.; J. SCHEFFER, A. M., Hagerstown, Md.; M. E. GRABER, Student, Heidelberg University, Tiffin, O.; and G. B. M. ZERR, A.M., Ph.D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

We have here the epicycloid and hypocycloid. The equation of the former is

$$x=(R+r)\cos\phi-r\cos\frac{R+r}{r}\phi,\text{ and } y=(R+r)\sin\phi-r\sin\frac{R+r}{r}\phi.$$

$$\therefore \frac{dx}{d\phi}=-(R+r)\sin\phi+(R+r)\sin\frac{R+r}{r}\phi.$$

$$\frac{dy}{d\phi}(R+r)\cos\phi-(R+r)\cos\frac{R+r}{r}\phi.$$

But $\dfrac{ds^2}{d\phi^2}=\dfrac{dx^2}{d\phi}+\dfrac{dy^2}{d\phi^2}=4(R+r)^2\sin^2\dfrac{R}{2r}\phi.$

$\therefore$ Length of curve between cusp and cusp

$$=2\int_0^{2+r/R}(R+r)\sin\frac{R}{2r}\phi=\frac{8r}{R}(R+r).$$

In the case of the hypocycloid we have in its equation only to put $-r$ in lieu of r, and by proceeding in the same way we obtain the length of the curve

$$=\frac{8r}{R}(R-r).$$

AVERAGE AND PROBABILITY.

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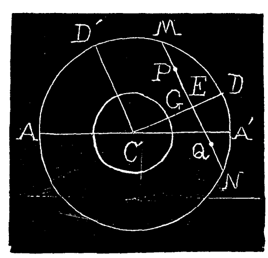
86. Proposed by L. C. WALKER, Assistant in Mathematics in Leland Stanford, Jr., University. Palo Alto, Cal.

Two points are taken at random in a circular annulus formed by two concentric circles. Find the chance that the straight line joining the points will not cut the inner variable circle.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let P, Q be the two random points, MN the chord through P, Q

Let $AC=r$, $CE=w$, $MQ=x$, $PQ=y$, $\angle ACD'=\theta$. CG, the radius of the variable circle $=u$, $p=$required chance.



An element of the circle at Q is $dwdx$, at P $yd\theta dy$. The limits of u are 0 and r; of w, r and u; of x, $2\sqrt{(r^2-w^2)}$ and 0; of y, 0 and x and doubled; of θ, 0 and 2π.

$$\therefore p=\frac{\dfrac{2}{\pi^2r^4}\displaystyle\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x\int_0^{2\pi}dudwdxydyd\theta}{\displaystyle\int_0^r du}$$

$$=\frac{4}{\pi r^5}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}\int_0^x dudwdxydy$$

$$=\frac{2}{\pi r^5}\int_0^r\int_u^r\int_0^{2\sqrt{(r^2-w^2)}}x^2\,dudwdx$$

$$=\frac{16}{3\pi r^5}\int_0^r\int_u^r(r^2-w^2)^{\frac{3}{2}}\,dudw$$

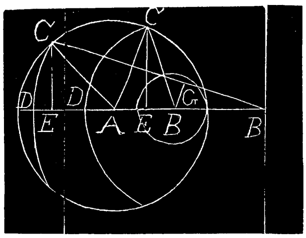
$$=\frac{1}{3\pi r^5}\int_0^r\left[3\pi r^4-10r^2u\sqrt{(r^2-u^2)}+4u^3\sqrt{(r^2-u^2)}-6r^4\sin^{-1}\frac{u}{r}\right]du$$

$$=\frac{16}{15\pi}=.3395.$$

87. **Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.**

Find the mean distance of a random point in a sphere from a point, (1) within, (2) without the sphere.

Solution by the PROPOSER.



Let B be the given point. $AB=c$, $AC=a$, $AD=\pm x$.

Then $DB=c-x=BC$, $CE=\sqrt{(a^2-AE^2)}$
$=\sqrt{[(c-x)^2-(c-AE)^2]}$.

$$\therefore\ AE=\frac{(a^2+2cx-x^2)}{2c},$$

$$DE=AE-x=\frac{(a^2-x^2)}{2c}.$$

Area of film $CDM=2\pi DB.DE=2\pi(c-x)(a^2-x^2)/2c$.

Also let $BG=y$. Then the area of the surface of this sphere is $4\pi y^2$.

$$(1).\ \text{Distance } D=\frac{\int_{-a}^{2c-a}2\pi DB^2.DEdx+\int_0^{a-c}4\pi y^3\,dy}{\frac{4}{3}\pi a^3}$$

$$\therefore\ D=\frac{3}{4a^3c}\int_{-a}^{2c-a}(c-x)^2(a^2-x^2)dx+\frac{3}{a^3}\int_0^{a-c}y^3\,dy.$$

$$\therefore\ D=\tfrac{3}{4}a+\frac{c^2}{2a}-\frac{c^4}{20a^3}.$$

$$(2).\ D_1=\frac{\int_{-a}^{a}2\pi DB^2.DEdx}{\frac{4}{3}\pi a^3}=\frac{3}{4a^3c}\int_{-a}^{a}(c-x)^2(a^2-x^2)dx.$$

$$\therefore\ D_1=c+(a^2/5c).$$

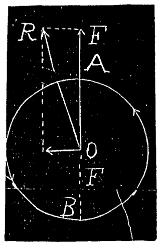
MISCELLANEOUS.

77. **Proposed by T. E. COLE, Columbus, Ohio.**

It is said that a base-ball pitcher throws curves. Give a scientific explanation of how it is done.

Solution by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.

When a ball pitcher wishes to throw a curve, in addition to the onward motion given, he sets the ball to revolving rapidly about its own axis. Now the resistance the air offers to a moving body depends upon its velocity, and the only resistance besides gravity which the ball encounters in its motion is this resistance of the air. Consider the ball as revolving from right to left as in the figure, and let us regard it as two bodies, the line of division being the line of its own motion onward. The particles of the right half of the ball are moving around either directly in the same direction as the onward motion, or in oblique directions; but these oblique velocities may all be resolved into two velocities. One parallel to and in the same direction OF, and one perpendicular to OF, which last has no effect on the onward velocity. Consequently the particles of the right half have, in addition to the onward velocity of the ball, a velocity



due to their revolution about O, this increase of velocity depending upon the distance of a particle from the center. Now in exactly the same way it can be shown that the velocity of particles on the left is decreased in proportion as that of those on the right is increased. Hence the resistance of the air offered to particles on the sight is slightly greater than that offered on the left. The result is that in effect a backward force is brought into play on the right which does not act on the left. Resolve this force parallel and perpendicular to OF. That parallel to OF does not effect the direction of motion. But combining that component perpendicular to F with F itself, we get that the total effect is to change the direction of the onward velocity, and since both the onward velocity and the revolution of the ball about its axis change at every instant of time, this change in direction must vary at any instant, and hence the ball moves in a curved line to which OR is a tangent.

This is merely a case of constrained motion as the above shows.

Also solved by *ALOIS F. KOVARIK.*

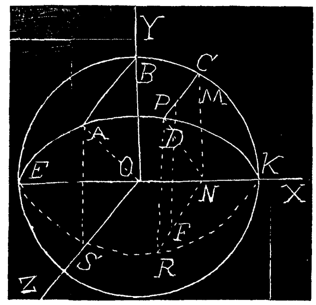
78. **Proposed by WALTER H. DRANE, A. M., Graduate Student, Harvard University, Cambridge, Mass.**

The center of a regular polygon of n sides moves along a diameter of a given circle, the plane of the polygon being perpendicular to the diameter, and its magnitude varying in such a manner that one of its diagonals always coincides with a chord of the circle; find the surface and the volume generated, and thence deduce the formulae for the surface and the volume of a sphere.

Solution by the PROPOSER.

Let AB be a side of the polygon in an initial position ; CD the side after

it has moved an infinitesimal distance ON. Then $ABCD$ is an elementary portion of $1/n$th the required area, and the prism $AOB-DNC$ an elementary portion of $1/n$th the required volume. The equation of the plane $ADKO$ is $z=\cot(\pi/n).y$. Let the equation of the sphere of which the circle KBE is a great circle be



$$x^2+y^2+z^2=a^2.$$

Combining the equation of the plane and sphere gives $x^2+\sec^2(\pi/n)z^2=a^2$, which is the equation of $ESRK$, the projection on the plane xz of the curve of intersection of the plane and sphere. Now let P be any point in the area. $CN=\sqrt{(a^2-x^2)}$. $CM=MP\tan MPC=MP\tan(\pi/n)=z\tan(\pi/n)$.

$PF=y$; $CN=CM+PF$ or $\sqrt{(a^2-x^2)}=z\tan(\pi/n)+y$, which is the equation of the surface.

$$\frac{A}{2n}=\int\frac{\sqrt{\left[\left(\frac{df}{dx}\right)^2+\left(\frac{df}{dy}\right)^2+\left(\frac{df}{dz}\right)^2\right]}}{\frac{df}{dy}}$$

$$=\int_0^a\int_0^{\cos(\pi/n)\sqrt{(a^2-x^2)}}\frac{\sqrt{[a^2\sec^2(\pi/n)-x^2\tan^2(\pi/n)]}}{\sqrt{[a^2-x^2]}}dzdx$$

$$\cos(\pi/n).\int_0^a\sqrt{[a^2\sec^2(\pi/n)-x^2\tan^2(\pi/n)]}dx$$

$$=\frac{\sin(\pi/n)}{2}\left[a^2\cot(\pi/n)+(\pi a^2/n)\operatorname{cosec}^2(\pi/n)\right].$$

$$\therefore\ A=na^2\cos(\pi/n)+\frac{\pi a^2}{\sin(\pi/n)}.$$

Evaluating this when $n=\infty$, by reducing to a common denominator, and then differentiating both numerator and denominator four times, this becomes $4\pi a^2$, the area of a sphere.

For volume we have,

$$V/2\pi=\sin(\pi/n)\cos(\pi/n)\int_0^a(a^2-x^2)dx=\tfrac{2}{3}a^3\sin(\pi/n)\cos(\pi/n).$$

$$\therefore\ V=\tfrac{4}{3}na^3\sin(\pi/n)\cos(\pi/n)=\tfrac{2}{3}na^3\sin(2\pi/n).$$

Throwing this in the typical form, and evaluating when $n=\infty$ by differentiating twice, gives $\frac{4}{3}\pi a^3$, the volume of a sphere.

Also solved by *G. B. M. ZERR*.

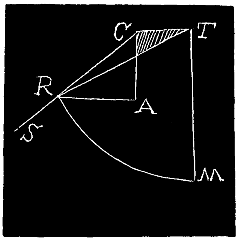
79. Proposed by S. HART WRIGHT, M. D., A. M., Ph. D., Penn Yan, N. Y.

In latitude 42° 30′ N.$=\lambda$, a tree 100 feet long$=\alpha$, leans in the direction S. 60° W.$=\beta$, with an angle of elevation with the level ground, of $30°=\gamma$. The sun's declination being 1° 36′ 24″ N.$=\delta$, in what direction will the shadow of the tree point, when the sun is on the meridian ?

I. Solution by the PROPOSER.

Let $TR=100=a$, be the tree, the top at R, TM a south line from T.

Draw a radii from R to A, and AT will be a sub-projection of RT, on the ground, the triangles TAR and CAR being vertical. Angles MTR, and $MTA=60°$ each$=\beta$, the triangle CAT being horizontal. Angle $ATR=30°=\gamma$, being the elevation of TR.



The shadow of R made by the sun at S must be cast north of A, and appear on the ground at C, and all the termini of shadows of all other points from R to T must be cast and visible on the line CT, as the sun is practically south of all points on the tree.

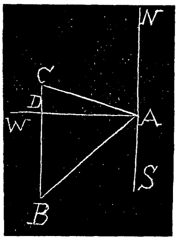
Angle $ACR=90°-\lambda+\delta=49°\ 6'\ 24''=c=$the sun's meridian altitude. $AR=a\sin\gamma=50$ feet$=e$, $AC=e\cot c$ $=43.3013$ feet$=\delta$, $AT=a\cos\gamma=86.6025$ feet$=b$.

Whatever angle CT makes with TM, counted from the south westward, will give the direction of the shadow.

CA must be parallel with TM. $\therefore$ Angle $CAT=ATM=60°$. Then in triangle ACT we have given AC and AT, and the included angle of 60°, to find the angle CTA, which, by Plane Trigonometry, is easily found to be 30°, ACT becoming in this particular case a right angle. Adding the angles CTA and ATM gives $MTC=90°$, and therefore the shadow points west.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Let NS be a north and south line on the ground, AB the projection of the tree upon the ground, AW the west line on the ground, BL perpendicular to AW, AC the shadow, $\angle NAC=\alpha$, and $\angle BAS=\beta$. $AB=a\cos\gamma$, $BD=a\cos\gamma\cos\beta$.



Since $90°-\lambda+\delta$ is the meridianal altitude we have BC $=a\sin\gamma\tan(\lambda-\delta)$.

$\therefore\ CD=a\sin\gamma\tan(\lambda-\delta)-a\cos\gamma\cos\beta$.

$$\therefore\ \cot x=\frac{CD}{AD}=\frac{a\sin\gamma\tan(\lambda-\delta)-a\cos\gamma\cos\beta}{a\cos\gamma\sin\beta}$$

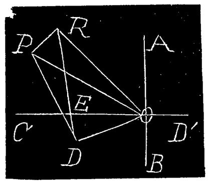
$$\therefore\ \cot x=\frac{\tan\gamma\tan(\lambda-\delta)}{\sin\beta}-\cot\beta.$$

Putting $\dfrac{\tan\gamma\tan(\lambda-\delta)}{\sin\beta}=\cot\varepsilon$, we get $\cot x=\dfrac{\sin(\beta-\varepsilon)}{\sin\beta\sin\varepsilon}$

For the given numerical values, $\gamma=30^\circ$, $\beta=60^\circ$, $\lambda-\delta=40^\circ\ 53'\ 36''$, we get $\varepsilon=60^\circ$. $\therefore \beta-\varepsilon=60^\circ$. $\therefore \cot x=0$. $\therefore x=90$. Consequently the shadow falls due west.

III. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $OP=100$ feet $=a$ be the tree, AB the meridian, CD' the parallel of latitude. OD the projection of the tree on the plane, OR the shadow, $\angle POD=30^\circ=\gamma$, $\angle DOB=60^\circ=\beta$, $\angle PRD=$sun's altitude$=\frac{1}{2}\pi-(\lambda-\delta)$, $\angle RDO=\beta$, $\angle DOE=\frac{1}{2}\pi-\beta$, $\angle PDO=\angle PDR=\angle DEO=\frac{1}{2}\pi$.



$\therefore PD=a\sin\gamma=50$ feet, $DO=a\cos\gamma=50\sqrt{3}$.

$\therefore DO=86.60$ feet, $DE=DO\cos\beta=43.30$ feet, $DR=PD\tan(\lambda-\delta)=a\sin\gamma\tan(\lambda-\delta)=43.30$ feet. $\therefore DE=DR$ and E and R coincide.

$\therefore$ The shadow is due west.

Also solved by *EDMUND FISH* and *A. H. BELL*, and *H. C. WHITAKER*.

80. Proposed by the late SYLVESTER ROBINS, North Branch, N. J.

Exhibit ten initials in that infinite series of integral, rational rhombuses wherein the area of every term is one unit less than the square of its side.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

Let $2a$ and $2b$=the respective diagonals of a rhombus. As the diagonals of a rhombus bisect each other at right angles, we then have, side of rhombus$=\sqrt{(a^2+b^2)}$, and area$=2ab$.

$\therefore$ From condition of problem, $2ab=a^2+b^2-1$, or $a^2-2ab+b^2=1$.

Whence $a-b=\pm1$.

$\therefore$ *The side of rhombus must be the hypothenuse of a right triangle whose legs are consecutive integers.*

Several methods for finding successive right triangles of this kind are given in The American Mathematical Monthly, Vol. IV, No. 1, pages 24—27.

Whence we find, for the first ten integral, rational rhombuses, the respective—

Diagonals,		*Sides,*	and *Areas.*
$2a$	$2b$	$\sqrt{(a^2+b^2)}$	$2ab=a^2+b^2-1$
8	6	5	24
42	40	29	840
240	238	169	28560
1394	1392	985	970224
8120	8118	5741	32959080
47322	47320	33461	1119638520
275808	275806	195025	38034750624
1607522	1607520	1136689	1292061882720
9369320	9369318	6625109	43892069261880
54608394	54608392	38613965	1491038293021224

The general formula for finding sides is $6S_{n-1}-S_{n-2}=S_n$.

Also solved by *A. H. BELL*, *CHAS. C. CROSS*, and *G. B. M. ZERR*.

PROBLEMS FOR SOLUTION.

ALGEBRA.

119. Proposed by HARRY S. VANDIVER, Bala, Montgomery Co., Pa.

Given $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{3\times5} + \frac{17x^7}{3^2\times5\times7} + \frac{62x^9}{3^2\times5\times7\times9} \ldots$

Find the general term and interval of convergence of this series.

120. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A hollow sphere has within it a solid sphere; a quantity of water equal to $1/m$ of the capacity of the hollow sphere is poured in and just covers the solid sphere. Prove that there are two solid spheres, either of which answers the conditions; also find the maximum value $1/m$, beyond which the question is not possible.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

CALCULUS.

111. Proposed by G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

(a). Find the dimensions of a cup, capacity c, in the form of a frustum of a pyramid regular, of n faces, so that its internal surface is a minimum.

(b). Find the dimensions of a cup, capacity c, in the form of a frustum of a hyperboloid or of a paraboloid, whichever it is, so that its internal surface is a minimum.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

MISCELLANEOUS.

91. Proposed by ARTEMAS MARTIN, LL. D., U. S. Coast and Geodetic Survey Office, Washington, D. C.

The following sides and area are given for a rational triangle in the table of rational scalene triangles on page 167 of Dr. Halsted's "Metrical Geometry" (Boston, 1881), viz.: sides, 21, 61, 65; area, 420. The same sides and area are given in Septimus Tebay's "Mensuration" (London and Cambridge, 1868), in a table on page 113.

The sides of this triangle can not all be correct because they are all *odd*.

Assuming that the *area* given is correct, it is required to determine the error in the sides.

92. Proposed by J. T. COLE, Columbus, Ohio.

A staff a=60 feet high, casting a shadow on a horizontal plane due north b=20 feet long, falls due northeast. Find the area covered by the shadow.

*** Solutions of these problems should be sent to J. M. Colaw not later than July 10.

NOTE ON THE SOLAR ECLIPSE OF MAY 28TH.

This eclipse was witnessed by the editor and his friend, Mr. Abel, from the campus of Tulane University of New Orleans, La. The atmospheric and meteorological conditions for observing this interesting phenomenon was most favorable. In the morning, at sun rise, the eastern sky was skirted with clouds, but by the time the first contact occurred, these had sunk nearer the horizon so as to leave the sun in a perfectly clear sky. First contact occurred at 6:25 A. M.; the second at 7:30; the third at 7:31 minutes 12 seconds, and the fourth at 8:43. Thus the period of totality was 1 minute and 12 seconds. Such a short period of totality did not give much time for the observation of those interesting phenomena attending an eclipse, viz., the "shadow bands" or "fringes," the corona, prominences, etc.

Prof. Ayers of the Tulane University had been making extensive preparations for the event for several months and during the time of the eclipse, took every precaution to eliminate as far as possible all sources of error. To this end it was ordered that visitors should not be allowed to approach the observatory nearer than 100 yards. This precaution was taken, since the motion, caused by the moving about of the visitors on the shallow crust of the earth here floating on a watery bed would be easily transmitted to the observatory and thus effect the instruments used in making observations. Professors Hume, Fulton, and Johnson, of the University of Mississippi, brought their astronomical instruments with them to Tulane University and coöperated in their observations with the Tulane University men.

It was through the courtesy of Prof. Hume that the editor was admitted to the sacred precincts of that eager and enthusiastic band of scientists, and through the courtesy of Professor Johnson was also permitted to take a view of the eclipsed sun with the instrument with which he was working.

The darkness was not as intense as one might suppose and not as dark as on previous occasions. The corona was very beautiful, though perhaps not so beautiful as has been seen during previous eclipses. Instead of streaming out on all sides, radiant filaments, beams, and sheets of pearly light, forming an irregular stellate halo. there appeared this time only a broad band of light lying in the direction in which the moon crossed the sun. The corona, therefore, had a marked resemblance to the corona of 1867. The "shadow bands" were quite clear to be seen at the time of totality. Whether a satisfactory explanation of their origin and nature has been discovered remains to be seen. Mr. Abel made several photographic exposures, but with what success is not yet known. Thorough and detailed accounts of this eclipse together with the results of the numerous observations made throughout the country, of course, will be given in the various Astronomical journals.

May 29, 1900.

EDITORIALS.

Mr. Sylvester Robins, of North Branch, N. J., one of our valued contributors, died suddenly and very unexpectedly, on Wednesday evening, April 25.

Mr. T. M. Putnam, Instructor in Mathematics in the University of Texas, has applied for a leave of absence to accept a Fellowship in the University of Chicago.

Dr. L. E. Dickson has resigned his position as Associate Professor of Mathematics in the University of Texas, to accept a call to the University of Chicago.

The B. F. Johnson Publishing Company, Richmond, Va., have in press a new two-book series of Arithmetics, by J. M. Colaw, Associate Editor of the MONTHLY, and J. K. Ellwood, Principal of the Colfax School, Pittsburg.

Thomas Craig, Ph. D., Professor of Pure Mathematics at Johns Hopkins University, a distinguished and widely-known mathematician, died in Baltimore, on May 8. Death was due to heart trouble. His wife and two daughters survive him. Professor Craig was born in Pittston, Pa., in the year 1855, and was in the very prime of life. He graduated from Lafayette College as a civil engineer in 1875, and took the degree of Doctor of Philosophy at Johns Hopkins in 1878. From his connection with the University, Professor Craig began his work in mathematics, which he continued to the time of his death, having borne a most important part in the work of the mathematical department. For many years he was editor of *The American Journal of Mathematics*. His more important works include "A Treatise on Linear Differential Equations," "A Treatise on Projections," and "Motions of Fluids." At the time of his death he was engaged on a new work entitled "Advanced Theory of Surfaces." Professor Craig was a frequent contributor to the different mathematical journals.

BOOKS AND PERIODICALS.

Rational Elementary Arithmetic. By H. H. Belfield, A. M., Ph. D., Director of the Chicago Manual Training School, and Sarah C. Brooks, Supervisor of Primary Grades, St. Paul, Minn. 268 pages. Price, 45 cents. Chicago: Scott, Foresman and Company. 1899.

This book contains an abundance of concrete work and graphic illustration. It is rich in suggestion and in thought-arousing matter. The book in the hands of an intelligent and skillful teacher should produce excellent results, but we should hesitate before placing it in the hands of an untrained or "unsupervised" teacher. J. M. C.

Bailey-Wiemer Series—First and Second Books in Arithmetic. By F. M. Wiemer, Principal of First District School, Milwaukee, Wis., assisted by M. A.

Bailey, A. M., Professor of Mathematics, Kansas State Normal School. 96 and 176 pages. New York: American Book Company. 1899.

These books abound in figures and indicated operations, but are deficient in objective work. If figure processes were the *end*, instead of the *means* to the end, these little volumes would be especially meritorious. However, with an expert teacher to supply the needed concrete work, they should give good results. J. M. C.

Manual of Experimental Physics. For Secondary Schools. By Fred R. Nichols, Charles H. Smith, and Charles M. Turton, Instructors in Physics in Chicago High School. 8 vo., cloth, 324 pages. Chicago: Ginn & Co.

This book embraces a thorough course in Laboratory work in Physics for High School and Academies. The work aims to make laboratory work inductive as far as possible. B. F. F.

Nine Ninety-Nine Graded Problems in Arithmetic. By Fred V. Lester, A. M., Superintendent of Schools, Ticonderoga, N. Y. 94 pages. Syracuse: C. W. Bardeen. 1899.

This little book gives a serviceable collection of problems for review work.

J. M. C.

Lessons in Arithmetic—Primary Number. By C. L. Howard, of the Columbia School, St. Louis, 72 pages. Price, 25 cents. St. Louis: W. S. Bell & Son. 1899.

The author puts forth this small volume as the result of his experience and the growth of trial. It seems to be well suited for use at the stage when children are acquiring their first knowledge of numbers and number relations. J M. C.

Graded Lessons in Arithmetic. Books II—VIII. By Wilbur F. Nichols, A. M., Principal Hamilton Street School, Holyoke, Mass. Price, 25 cents each. Boston: Thompson, Brown & Co. 1897-1899.

The plan of the author is to present the elements of many topics in the lower grades, and then to make the work in each topic more difficult through the subsequent grades. These books furnish a large number of well-graded examples of great variety on all the different topics. An examination in detail reveals many good features. J. M. C.

Gay's Problems in Arithmetic. Books I and II. By George E. Gay, Superintendent of Schools, Malden, Mass. Boston: Benj. H. Sanborn & Co. 1898.

Each of these books contains 1,000 problems designed for written work. The problems are progressive and practical, and are presented in an attractive form. J. M. C.

First Steps in Arithmetic. By Ella M. Pierce, Supervisor of Primary Grades, Public Schools, Providence, R. I. 160 pages. Price, 36 cents. Boston: Silver, Burdett & Co. 1899.

Twenty is the extreme of the numbers taught. The ordinary tables of measurement are freely used. The book is intended for beginners, and is well adapted to its end.

J. M. C.

The Werner Arithmetic. Book III. By Frank H. Hall. 282 Pages. Chicago: Werner School Book Company. 1898.

In Books I and II of this series, classification is made subordinate to gradation, but in this book the topical arrangement is given. The number of topics has been wisely reduced. The book is carefully written, and contains practical problems in great variety.

The introduction of the elements of algebra aad geometry is to be commended, but the systematic recurrence of these subjects in parts of the text, where they do not serve to illuminate the arithmetical treatment, detracts somewhat from the general excellence of the work. J. M. C.

Rand-McNally Primary Arithmetic. Revised Edition, 1899. By Edwin C. Hewett, LL. D., Ex-President of the Illinois State Normal University. 268 pages. Price, 30 cents. Chicago: Rand, McNally & Co.

This Primany book is unusually full in the early steps, and gives a good preparation for the advanced grades. The directions and suggestions to teachers are full and generally to the point. J. M. C.

Graded Work in Arithmetic. Fifth Book. By S. W. Baird, Principal Franklin Grammar School, Wilkesbarre, Pa. 356 pages. Price, 65 cents. New York: American Book Company. 1899.

This book completes a well-graded series, and is intended for grammar school grades. There are no rules, and a minimum of definitions and explanations. The supply of practical problems is abundant and varied. One chapter is devoted to Algebraic equations. J. M. C.

Lippincott's Arithmetics, Mental, Elementary and Practical. By J. Morgan Rawlins, A. M. Prices, 35 cents, 40 cents, and 65 cents, respectively. Philadelphia: J. M. Lippincott Company. 1899.

The aim of these books is "to prepare students for practical life by the development of thought power, while making them masters of the mechanical processes." The advanced book seems to be overburdened with definitions, principles, analyses, synthesis, rules, etc., and several of the topics included could well have been omitted as having no utilitarian value. The problems are fresh, well-arranged, and adequate in number. The first forty pages of the Elementary book are especially well suited to the needs of the beginner. J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single numbers, 25 cents. The Review of Reviews Co., New York.

Dr. Albert Shaw describes "Paris and the Exposition of 1900" in his magazine, the *Review of Reviews*, for June. Dr. Shaw regards Paris itself, the typically modern city, as an inseparable part of the great fair. So far from complaining of the incompleteness of the Exposition in the opening month, as many visitors have, Dr. Shaw welcomed the opportunity to see so many of the wonders of the fair in the making. His article is by far the most discriminating estimate of the real value of the Paris show that has been published on this side of the Atlantic.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single numbers, 10 cents. Funk and Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

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ON A DETERMINANT EACH OF WHOSE ELEMENTS IS THE PRODUCT OF k FACTORS.

By PROF. W. H. METZLER, Syracuse University, Syracuse, N. Y.

1. In Muir's *Theory of Determinants*, page 117, the following example (slightly modified) is given without comment :

$$\begin{vmatrix} h_1m_1 & h_1n_1 & k_1x_1 & k_1z_1 \\ h_2m_1 & h_2n_1 & k_2x_1 & k_2z_1 \\ p_1m_2 & p_1n_2 & q_1x_2 & q_1z_2 \\ p_2m_2 & p_2n_2 & q_2x_2 & q_2z_2 \end{vmatrix} = \begin{vmatrix} h_1 & k_1 & 0 & 0 \\ h_2 & k_2 & 0 & 0 \\ 0 & 0 & p_1 & q_1 \\ 0 & 0 & p_2 & q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1 & 0 & m_2 & 0 \\ n_1 & 0 & n_2 & 0 \\ 0 & x_1 & 0 & x_2 \\ 0 & z_1 & 0 & z_2 \end{vmatrix}$$

$$= \begin{vmatrix} h_1k_1 \\ h_2k_2 \end{vmatrix} \cdot \begin{vmatrix} p_1q_1 \\ p_2q_2 \end{vmatrix} \cdot \begin{vmatrix} m_1n_1 \\ m_2n_2 \end{vmatrix} \cdot \begin{vmatrix} x_1z_1 \\ x_2z_2 \end{vmatrix} .$$

From this it is seen that if we take n_1 determinants

$$| b_{1n_2}^{(n_2, j_1)} | , \; (j_1=1, 2....n_1),$$

of order n_2 and form a determinant of order $n=n_1n_2$ by arranging them along the principal diagonal as in the first factor of the right-hand member of the first equation in the above example, all the other elements being zeros, and n_2 determinants

$$| b_{1n_1}^{(n_1, j_2)} | , \; (j_2=1, 2....n_2),$$

of order n_1 and form another determinant of order n in which the elements in the $(\alpha n_2+\beta)$th column of the βth n_1 rows are the elements of the $(\alpha+1)$th column of

$$| b_{1n_1}{}^{(n_1, \beta)} |, \ (\alpha=0, 1 \ldots . n-1), \ (\beta=1, 2 \ldots . n_2),$$

all the other elements being zeros, the product of these two determinants of the nth order is a determinant of the nth order whose elements are products of two factors.

It is also seen that the elements of $| b_{1n_2}{}^{(n_2, j_1)} |$ are found in all the columns, but in the j_1th n_2 rows only of the product, and that the elements of $| b_{1n_1}{}^{(n_1, j_2)} |$ are found in all the rows, but in the (αn_2+j_2)th columns only of the product.

2. We may now form two determinants of the nth order $(n=n_1n_2n_3)$ precisely as in Art. 1, first by taking n_3 determinants of order n_1n_2 each of whose elements is the product of two factors as there found, and second by taking n_1n_2 determinants

$$| b_{1n_3}{}^{(n_3, j_3)} |, \ (j_3=1, 2 \ldots . (n/n_3)$$

of order n_3, and the product of these two determinants of the nth order will be a determinant of the nth order each of whose elements is the product of three factors, one factor an element from a determinant of each of the three orders.

Continuing this process we arrive at a determinant of the order $n=n_1n_2 \ldots n_k$, each of whose constituents is the product of k factors, one factor an element from a determinant of each of the k given orders $n_1, n_2 \ldots . n_k$

3. Let $\left\{ \frac{\alpha}{\beta} \right\}$ denote the greatest integer in $\frac{\alpha}{\beta}$, $R\frac{\alpha}{\beta}$ denote the remainder on dividing α by β,

$$| b_{1n_g}{}^{(n_g, j_g)} |, \ (g=1, 2 \ldots . k), \ (j_g=1, 2 \ \ldots n/n_g),$$

denote the n/n_g determinants of order n_g. Then if $n=n_1n_2 \ldots . n_k$ we have the theorem

$$A= | a_{1n} | = \Pi | b_{1n_g}{}^{(n_g, j_g)} |, \ (g=1, 2 \ldots . k), \ (j_g=1, 2 \ldots . n/n_g) \ldots . (1).$$

Where the element in the xth row and yth column of A,

$$a_{xy}=b_{R\frac{x-1}{n_1}+1, \left\{ \frac{y-1}{n/n_1} \right\}+1}^{\left(n_1, \left\{ \frac{x-1}{n_1} \right\}+1\right)}$$

$$\prod_{h=2}^{h=k} b \begin{pmatrix} n_h, \left\{\dfrac{x-1}{n_1 n_2 \ldots n_h}\right\} n_1 n_2 \ldots n_{h-1} + \left\{\dfrac{\dfrac{y-1}{n}}{n_1 n_2 \ldots n_{h-1}}\right\} + 1 \\ \left\{R \dfrac{\dfrac{\dfrac{y-1}{n}}{n_1 n_2 \ldots n_{h-1}}}{\dfrac{n}{n_1 n_2 \ldots n_h}}\right\} + 1, \left\{R \dfrac{\dfrac{x-1}{n_1 n_2 \ldots n_h}}{n_1 n_2 \ldots n_{h-1}}\right\} + 1 \end{pmatrix} \quad \ldots (2).$$

Each of the elements $b_{\alpha_g \beta_g}{}^{(n_g,\, i_g)}$ $(\alpha_g, \beta_g = 1, 2 \ldots n_g)$ occurs in $\dfrac{n}{n_g \ldots n_k}$ rows and in $\dfrac{n}{n_1 \ldots n_g}$ columns of A.

If we write $i_g = \lambda_g . n_1 n_1 \ldots n_{g-1} + \mu_g$, where

$$\left\{\lambda_g = 0, 1 \ldots \left(\frac{n}{n_1 \ldots n_g} - 1\right)\right\},$$

$$\{\mu_g = 1, 2 \ldots (n_1, n_2 \ldots n_{g-1})\},$$

and understand that when $g=1$, $n_1 n_2 \ldots n_{g-1} = 1$.

Then the element $b_{\alpha_g \beta_g}{}^{(n_g,\, \lambda_g n_1 \ldots n_{g-1} + \mu_g)}$ occurs in the $(\lambda_g n_g + \alpha_g)$th $n_1 n_2 \ldots n_{g-1}$ rows, and in the $\{(\mu_g - 1) n_g + \beta_g\}$th $\dfrac{n}{n_1 \ldots n_g}$ columns of A.

4. If all the determinants of the same order n_g, $(g=1, 2 \ldots k)$ are equal the theorem becomes

$$A = \Pi \mid b_{1 n_g}{}^{(n_g)} \mid^{i_g},\ i_g = n/n_g \quad \ldots (3).$$

If $n_1 = n_2 = n_3 = \ldots = n_k = m$, then $n = m^k$ and the theorem takes the form

$$A = \Pi \mid b_{1m}{}^{(h)} \mid,\ (h = 1, 2 \ldots k . m^{k-1}) \quad \ldots (4).$$

If all the determinants are of the same order and equal to each other then the theorem becomes

$$A = \mid b_{1m} \mid^{k . m^{k-1}} \quad \ldots (5).$$

After finding the general theorem in Art. 3, the special cases (3) and (5) were first made known to me by Prof. E. H. Moore, who discovered them before knowing of the general theorem, and who immediately made the generalization, using the different notation, on receiving the Muir reference.

December 5, 1898.

THE GEOMETRIC OLD AND NEW.

By DR. GEORGE BRUCE HALSTED.

1. Perhaps the newest geometry in English is by James Howard Gore, professor of mathematics in Columbian University, and published by Longmans, Green & Co., first edition December, 1898, second edition, 1899. Mr. Gore defines the straight in his §6, as follows: "A straight line is the shortest line between two points." But this presumes beforehand the measurement of all lines, while the soundest of geometries, Euclid, will not even attribute length to the simplest of curves, the circle, and our new mathematics knows of lines, real boundaries between two parts of a plane, to which the idea of length is inapplicable. Moreover, before the execution of any measurement there must be a measuring standard; but this is first given by the straight line, is in fact always *a sect*, a definite piece of a straight. Again the existence of a minimum is presumed; which is not evident; and of a single minimum; which is a subtle assumption.

In fact the operation of measuring a geometric magnitude we cannot effect, rigorously speaking, either for curves or curved surfaces. For, however little may be the parts of a curve, they do not cease to be curves, and consequently they cannot be compared with a sect by superposition or congruence; just as parts of a curved surface are not comparable with portions of a plane by superposition or congruence. A paradoxical assumption is in fact necessary, which destroys by itself the primitive idea of measurement, the application of a standard unit.

Thus the evaluation of the length of a curve represents not at all a measurement of the primary kind, and until explicit post-Euclidean assumptions have been made, we cannot even know what is to be meant by one line being shorter than another between the same two points.

2. Mr. Gore defines an angle in §30: "An angle is the difference in direction of two lines that meet." But the word 'direction' can only be given a meaning when the theory of parallels is presupposed. No one has ever been able to say when two different straights have the same direction without using parallelism. Direction, to be understood in any strict sense whatever, presupposes three fundamental geometric ideas, namely, straight line, angle, parallels.

3. In §52 in the last step of an attempted demonstration, we read: "this would give two straight lines joining P and P', *which is impossible.*" On the contrary, in Riemann's double elliptic geometry every two straight lines meet twice.

4. Under Parallel Lines, we read: "§59. Definition. Two straight lines are called Parallel when they lie in the same plane, and cannot meet, *nor approach each other*, however far they may be produced."

Mr. Gore's unfortunate interpolation is thus paraphrased in his next article: "Since parallel lines cannot approach each other, they are everywhere

equally distant from each other." This assumes that a line which is everywhere equidistant from a straight line must be itself straight, but in Bolyai's geometry this line is a curve, the well-known *equidistantial*, while in Lobachevski one of the first theorems is: "The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another."

In §62, in place of a proof for the fundamental theorem: "If two parallels are cut by a transversal the alternate angles are equal," Mr. Gore gives the flat *petitio principii*: "The lines AB and CD, being parallel, have the same direction. The lines EG and GH, being in one and the same straight line, are similarly directed. That is, the angles EGB and GHD have sides with the same direction; therefore the differences of their directions are equal." ! ! !

"§127. A circle is a plane figure"!

"§138. Two circumferences are tangent to each other when they are tangent to a straight line at the same point"! What about two circles with only one point in common, and the straight tangent to one at that point different from the straight tangent to the other at that point?

Page 64, "Up to the present time it has been assumed that any needful line or combination of lines could be drawn, and the question has not arisen as to the possibility of drawing these lines *with accuracy! ! !*

In order to show that any required combination of lines, angles, or parts of lines or angles fulfilled the required conditions, principles were needed long before they could be demonstrated"! Mr. Gore has perhaps never looked into a copy of Euclid. Beyond his three postulates, hypothetical constructions are neither necessary nor admissible.

There is one good purpose that this book may subserve, that is to show how absolutely essential is a knowledge of non-Euclidean geometry.

Austin, Texas, 1900.

EXPRESSION OF RIEMANN'S P FUNCTION AS A DEFINITE INTEGRAL.

By W. E. HEAL.

Consider the differential equation of Riemann's P function, namely,

$$(x-a)(x-b)(x-c)\frac{d^2y}{dx^2}+(x-a)(x-b)(x-c)\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}\right.$$

$$\left.+\frac{1-\gamma-\gamma'}{x-c}\right)\frac{dy}{dx}+\left(\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}\right.$$

$$\left.+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\right)y=0 \;\ldots\ldots (1),$$

where α, α', β, β', γ, γ' are real quantities and we have $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma'=1$.

Let $V=(x-a)^{\alpha}(x-b)^{\beta}(x-c)^{\gamma}u^{-(\alpha'+\beta'+\gamma')}$

$$(1-u)^{-(\alpha'+\beta+\gamma')}[(a-b)(x-c)+(b-c)(x-a)u]^{-(\alpha+\beta+\gamma)}.$$

Then $y_1=\int_0^1 Vdu$ is an integral of equation (1).

For substituting this value for y in the equation (1) the left member becomes

$$\int_0^1\Big[(x-a)(x-b)(x-c)\left\{\frac{(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}\right.$$

$$-\frac{(\alpha+\beta+\gamma)\left(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\right)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}$$

$$+\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right)\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)$$

$$\left.+\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)^2-\left[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\right]\right\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}\Big]Vdu.$$

Since $(x-a)(x-b)(x-c)\left\{\left(\frac{1-\alpha-\alpha'}{x-a}+\frac{1-\beta-\beta'}{x-b}+\frac{1-\gamma-\gamma'}{x-c}\right)\right.$

$$\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)+\left(\frac{\alpha}{x-a}+\frac{\beta}{x-b}+\frac{\gamma}{x-c}\right)^2$$

$$\left.-\left[\frac{\alpha}{(x-a)^2}+\frac{\beta}{(x-b)^2}+\frac{\gamma}{(x-c)^2}\right]\right\}$$

$$+\frac{\alpha\alpha'(a-b)(a-c)}{x-a}+\frac{\beta\beta'(b-c)(b-a)}{x-b}+\frac{\gamma\gamma'(c-a)(c-b)}{x-c}$$

$$=(\alpha+\beta+\gamma)[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c],$$

this result may be written,

$$(\alpha+\beta+\gamma)\int_0^1\Big[(x-a)(x-b)(x-c)\left\{\frac{(\alpha+\beta+\gamma+1)[(a-b)+(b-c)u]^2}{[(a-b)(x-c)+(b-c)(x-a)u]^2}\right.$$

$$-\frac{\left(\frac{1+\alpha-\alpha'}{x-a}+\frac{1+\beta-\beta'}{x-b}+\frac{1+\gamma-\gamma'}{x-c}\right)[(a-b)+(b-c)u]}{[(a-b)(x-c)+(b-c)(x-a)u]}\Big\}$$

$$+[(\alpha+\beta+\gamma+1)x-(\alpha'+\beta+\gamma)a-(\alpha+\beta'+\gamma)b-(\alpha+\beta+\gamma')c]\,Vdu$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)$$

$$\int_0^1 \frac{\{(\alpha+\beta+\gamma')(a-b)(x-c)+(\alpha'+\beta+\gamma)(b-c)(x-a)u^2 -[(a-b)(1+\alpha-\alpha')(x-c)+(b-c)(1+\gamma-\gamma')(x-a)]u\}\,Vdu}{[(a-b)(x-c)+(b-c)(x-a)u]^2}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\int_0^1 \frac{d}{du}\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}$$

$$=-(\alpha+\beta+\gamma)(a-b)(b-c)(c-a)\left\{\frac{u(1-u)V}{[(a-b)(x-c)+(b-c)(x-a)u]}\right\}_0^1$$

which is identically zero if the integral has a meaning. That the integral may not become infinite at the limits we must have

$$1-(\alpha'+\beta'+\gamma)=\alpha+\beta+\gamma'>0,$$
$$1-(\alpha'+\beta+\gamma')=\alpha+\beta'+\gamma>0.$$

It is also clear that

$$y_2=\int_0^{-\infty} Vdu, \quad y_3=\int_1^{+\infty} Vdu,$$

also satisfy the differential equation. For $y=y_2$ we must have

$$\alpha+\beta+\gamma'>0, \quad \alpha'+\beta+\gamma>0.$$

And for $y=y_3$,

$$\alpha+\beta'+\gamma>0, \quad \alpha'+\beta+\gamma>0$$

In equation (1) write

$$a=-1, \; b=1/\varepsilon, \; c=+1,$$
$$\alpha=\alpha'=\gamma=\gamma'=0, \; \beta=-n, \; \beta'=n+1, \; \lim\varepsilon=0,$$

and we find after some reductions the differential equation for zonal spherical harmonics, viz :

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{n(n+1)y}{1-x^2}=0.$$

(Craig's *Linear Differential Equations*, page 192.) In this case

$$y_1=\int_0^1\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_2=\int_0^{-\infty}\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}},$$

$$y_3=\int_1^{+\infty}\frac{(1-u)^n[(1-x)+(1+x)u]^n du}{u^{n+1}}.$$

In y_1 we must have n negative and numerically less than unity. For y_3 the values n may have are the same as for y_1. In y_2 we may have n any, negative, real number.

In equation (1) write

$$a=0,\ b=1,\ c=-1,$$
$$\alpha=\tfrac{1}{2}\{(k-1)+\sqrt{[4n(n+k-1)+(k-1)^2]}\},$$
$$\alpha'=\tfrac{1}{2}\{(k-1)-\sqrt{[4n(n+k-1)+(k-1)^2]}\},$$
$$\beta=\gamma=\tfrac{1}{4}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},$$
$$\beta'=\gamma'=\tfrac{1}{4}\{(2-k)+\sqrt{[4m(m+k-2)+(k-2)^2]}\},$$

and we have the differential equation,

$$\frac{d^2y}{dx^2}+\frac{2x^2+k-2}{x(x^2-1)}\frac{dy}{dx}+\frac{[n(n+k-1)(x^2-1)-m(m+k-2)x^2]y}{x^2(x^2-1)^2}=0 \ldots\ldots\ldots(3),$$

which transformed by the substitution $x=1/t$ becomes the equation for spherical harmonics of rank k, namely,

$$\frac{d^2y}{dt^2}-\frac{kt}{1-t^2}\frac{dy}{dt}+\frac{[n(n+k-1)(1-t^2)-m(m+k-2)]y}{(1-t^2)^2}=0 \ldots\ldots\ldots(4),$$

(Craig, page 195.)

The definite integral appears to be too complicated to be of much use except in special cases.

I. Let $k=1$.

$\therefore\ \alpha=n,\ \alpha'=-n,\ \beta=\gamma=\tfrac{1}{2}m,\ \beta'=\gamma'=\tfrac{1}{2}(1-m).$

Equation (4) becomes for this case,

$$\frac{d^2y}{dt^2}-\frac{t}{1-t^2}\frac{dy}{dt}+\frac{[n^2(1-t^2)-m(m-1)]y}{(1-t^2)^2}=0 \quad \ldots (5).$$

We have,

$$y_1=\int_0^1 (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_2=\int_0^{-\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du,$$

$$y_3=\int_1^{+\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2n-1)}.[2u-(1+t)]^{-(m+n)}du.$$

In all these integrals we have

$$n+\tfrac{1}{2}>0,$$

and in y_2, y_3 we must also have

$$m>n.$$

II, Let $k=2$.
$\therefore\ \alpha=n+1,\ \alpha'=-n,\ \beta=\gamma=\frac{1}{2}m,\ \beta'=\gamma'=-\frac{1}{2}m.$
Equation (4) becomes

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[n(n+1)(1-t^2)-m^2]y}{(1-t^2)^2}=0 \ldots . (6),$$

which is (Craig, page 194) the equation of the *associated function*, $P_{n,\,m}$, of the first kind, of degree n and order m.

We have

$$y_1=\int_0^1 (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_2=\int_0^{-\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du,$$

$$y_3=\int_1^{+\infty} (1-t^2)^{\frac{1}{2}m}[u(1-u)]^n.[2u-(1+t)]^{-(m+n+1)}du.$$

In these integrals we must have

$$n+1>0,$$

and y_2, y_3 we also have

$$m>n.$$

If in (6) we put $n=\mu-\frac{1}{2}$ we have,

$$\frac{d^2y}{dt^2}-\frac{2t}{1-t^2}\frac{dy}{dt}+\frac{[(\mu^2-\frac{1}{4})(1-t^2)-m^2]y}{(1-t^2)^2}=0\ldots.(7),$$

which is (Craig, page 194) the differential equation for Hicks' Toroidal functions.

We have for this case

$$y_1=\int_0^1(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_2=\int_0^{-\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

$$y_3=\int_1^{+\infty}(1-t^2)^{\frac{1}{2}m}[u(1-u)]^{\frac{1}{2}(2\mu-1)}.[2u-(1+t)]^{-\frac{1}{2}(2m+2\mu+1)}\,du,$$

where $\mu+\frac{1}{2}>0$, and in y_2, y_3 $m>\mu-\frac{1}{2}$.

FORECASTING THE CENSUS RETURNS.

By JAMES S. STEVENS, Professor of Physics, The University of Maine, Orono, Maine.

Now that the government of the United States is about to take another census, we occasionally see in the newspapers forecasts of the population. Sometimes these forecasts are mere guesses, but there is a method by which one can make these estimates scientifically, and if they fail to come out right it is the fault of the people rather than the method.

All physical laws may be divided into two classes—rational and empirical. The free fall of a body, the swinging of a pendulum, and most of the laws of heat and electricity illustrate the first class. If a body falls one space the first second it will fall three the second and five the third. These laws may easily be embodied into formulae which contain no arbitrary constants. On the other hand, such problems as the relation between temperature and depth below the

surface, and the gravitational acceleration over the earth's surface follow no fixed law. In order to express them at all, special formulae must be devised which contain constants whose values are to be determined.

It is obvious that the law of increase of population of a country belongs to this latter class. To construct an empirical formula a curve is plotted showing the relation between the various decades and their corresponding population. When this is drawn, unless it is too irregular, a mathematician can locate it among the curves with which he is familiar, and the equation of the curve is the empirical formula required. The census curve for the United States turns out to be a parabola whose equation is $p=S+Tx+Ux^2$ in which p is the population, x the number of the decade, and S, T, and U are unknown constants.

To determine these constants we first write down the population for the various decades. (These figures are from the World Almanac.)

1790	—	3.6
1800	—	5.3
1810	—	7.2
1820	—	9.6
1830	—	12.9
1840	—	17.1
1850	—	23.2
1860	—	31.4
1870	—	38.6
1880	—	50.2
1890	—	62.6

The population is expressed in millions and tenths of a million. Substituting these values in our formula we have:

$$\begin{aligned}
3.6&=S-T+U\\
5.3&=S+0+0\\
7.2&=S+T+U\\
9.6&=S+2T+4U\\
12.9&=S+3T+9U\\
17.1&=S+4T+16U\\
23.2&=S+5T+25U\\
31.4&=S+6T+36U\\
38.6&=S+7T+49U\\
50.2&=S+8T+64U\\
62.6&=S+9T+81U
\end{aligned}$$

These are called "observation equations." If we multiply each one in turn by the coefficients of S, T, and U respectively, and add the results, we have what are called "normal equations." They are as follows:

$$
\begin{aligned}
-3.6 &= -S + T - U \\
7.2 &= S + T + U \\
19.2 &= 2S + 4T + 8U \\
38.7 &= 3S + 9T + 27U \\
68.4 &= 4S + 16T + 64U \\
116.0 &= 5S + 25T + 125U \\
188.4 &= 6S + 36T + 216U \\
270.2 &= 7S + 49T + 343U \\
401.6 &= 8S + 64T + 512U \\
563.4 &= 9S + 81T + 728U \\
\hline
1669.5 &= 44S + 286T + 2023U
\end{aligned}
$$

$$
\begin{aligned}
3.6 &= S - T + U \\
7.2 &= S + T + U \\
38.4 &= 4S + 8T + 16U \\
116.1 &= 9S + 27T + 81U \\
273.6 &= 16S + 64T + 256U \\
580.0 &= 25S + 125T + 625U \\
1130.4 &= 36S + 216T + 1296U \\
1891.4 &= 49S + 343T + 2401U \\
3212.8 &= 64S + 512T + 4096U \\
5070.6 &= 81S + 729T + 6561U \\
\hline
12324.1 &= 286S + 2026T + 15334U
\end{aligned}
$$

When we multiply through by the coefficients of S (unity) the observation equations undergo no change and their sum is :

$$261\ 7 = 11S + 44T + 286U.$$

We have now three equations with three unknown quantities which we may solve in any manner we choose. The values obtained are :

$$
\begin{aligned}
S &= 6.08 \\
T &= 0.690 \\
U &= 0.622
\end{aligned}
$$

If we substitute these in the formula we get :

$p = 6.08 + 6.9 + 62.2$, or
$p = 75.2$ millions, which is the forecast for 1900.

If we neglect the curves of 1890 the formula becomes :

$$p = 4.97 + 0.873x + 0.581x^2.$$

If we use only those from 1820 to 1880 inclusive we get :

$$p=7.29-0.28x+0.689x^2.$$

For the year 1894 the first of the formula gave 64.5 and the second 65.5 ; while the population reported in the World Almanac was 66.7.

The second formula gives for 1900 a population of 73.4, while the first formula gives a result somewhat lower. It appears then, that the population is increasing more rapidly than the parabolic curve indicates, and that if anything our forecast of 73 4 is somewhat low. It must be understood of course that the above formulae are strictly anti-expansion and make no allowance for our new possessions.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the February Number.]

CHAPTER VI.

GEOMETRICAL ADDITION AND SUBTRACTION.

73. The general theory of the *Ausdehnungslehre* may be applied in such diverse sciences as geometry, mechanics, and logic. We proceed in this and the following chapter to apply it in geometry.

74. The concepts dealt with in geometry are the *point*, *line*, *surface*, and *solid*, which may or may not be fixed in position. For the sake of distinction a line whose length and direction are fixed but not its position is called a *vector*. (3). A portion of a plane whose direction and extent are fixed but not the position of the plane is called by analogy a *plane vector*.

75. As an introduction to the *Ausdehnungslehre* the addition and subtraction of vectors was treated in Chapter I. It is evident from what was given in that chapter that plane vectors may be added and subtracted in the same way as line vectors. One gets the parts whose sum is a given plane vector by projecting the given plane vector on the coordinate planes.

As we have already treated of the addition and subtraction of vectors, we proceed to apply the laws of addition and subtraction to points.

76. We will define a point as an infinitesimal portion of a line and denote it by p. When the point has position we will denote it by p_ρ, in which p denotes the point at the extremity of the radius vector ρ from the origin, O. Evidently a line, or plane, or solid may be located by means of a radius vector in the same way. (See 168.)

Grassmann defines a point as that which has position and uses a single letter as A to denote it. In what follows if the ρ's be cut out of the formulas and the p's be given the subscripts of the ρ's, Grassmann's expressions will result.

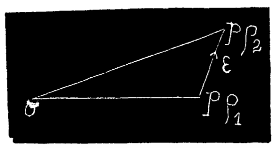
The reasons for using the complex symbol $p\rho$, instead of the simpler A, are : (1) Because the concept is complex and therefore for clearness should be represented by a complex symbol. (2) Since position is relative, for the proper representation of positions an origin is needed. (3) Because this notation shows plainly the relation which exists between point and vector analysis.

77. What we will call *unit* points all have the same (infinitesimal) unit length. This length as also that of the radius vector may be multiplied by any scalar, m. Thus $mp\rho$ denotes the point whose length or "weight," as it is called, is mp held in position by ρ, while $p(m\rho)$ denotes p held in position by $m\rho$, *i. e.*, m times the length of ρ.

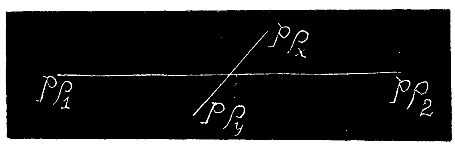
78. The difference between two unit points, since they can differ only in position, is a certain distance in a certain direction, *i. e.*, is a vector. (See 3.)

Thus $\quad p\rho_2 - p\rho_1 = \rho_2 - \rho_1 = \varepsilon$. (4).

Similarly, $\quad mp\rho_2 - mp\rho_1 = m\varepsilon$.

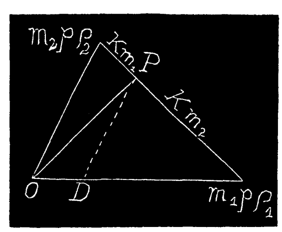


79. We next seek to find the sum of two or more points. What this sum is remains to be determined. Grassmann gives an investigation to show that the sum of two unit points is a point on the line joining them. We abridge this proof as follows : He begins by postulating (1) That whatever is true of one set of points is true of any congruent system wherever situated ; (2) That the fundamental laws of addition and subtraction (14) hold. Then he assumes that the sum of two points is some point.



Let, in the figure, $p\rho_1$ and $p\rho_2$ be any two unit points whose sum is sought. Suppose $p\rho_1 + p\rho_2 = p\rho_x$. Then revolving the whole figure in the plane of the paper through 180°, $p\rho_1$ coincides with $p\rho_2$, $p\rho_x$ with $p\rho_y$, and $p\rho_2$ with $p\rho_1$. Thus we get $p\rho_2 + p\rho_1 = p\rho_y$. But by 14, $p\rho_2 + p\rho_1 = p\rho_1 + p\rho_2$. Then $p\rho_x = p\rho_y$. This can only happen when they both coincide with the midpoint of the straight line joining the two given points.

80. Mechanics gives us a simpler and more general interpretation for the sum of two or more points. Let us regard the points as parallel (infinitesimal) forces whose magnitudes are represented by the weights of the points. The law for the addition of parallel forces gives a simple and consistent result. Thus



$$m_1 p\rho_2 + m_2 p\rho_2 = (m_1 + m_2) p \left(\frac{m_1\rho_1 + m_2\rho_2}{m_1 + m_2} \right),$$

i. e., the sum of the two weighted points is a point on the line joining them whose

weight is the sum of the weights of the two points and the extremity of whose radius vector divides the line joining the two given points into segments inversely proportional to the weights of these points.

That $\frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}$ is the vector to the point described is evident from elementary geometry. Thus regarding the two radii vectores going out from O as axes, it is easy to show by similar triangles that

$$OD=\frac{m_1}{m_1+m_2}\rho_1, \text{ and } DP=\frac{m_2}{m_1+m_2}\rho_2.$$

NOTE.—The letter k which appears on the figure above is a factor chosen such that km_1 and km_2 equal the segments designated by them.

81. *Generalizing the result of the last article, we have*

$$\Sigma m_1 p\rho_1=\Sigma m_1 . p\left(\frac{\Sigma m_1\rho_1}{\Sigma m_1}\right).$$

82. When $\Sigma m_1=0$ in the preceding result, the weight of the sum point is zero and the radius vector is infinite in length. To interpret this, we get the sum of all the points except one and then add this partial sum to the remaining point. In this way we obtain an expression similar to that of 78 where the result is a *vector*.

Hence $\Sigma m_1 p\rho_1$ is a VECTOR when $\Sigma m_1=0$, and a POINT when Σm_1 is not equal to 0.

Thus a point at infinity (of zero weight) is equivalent to a vector.

83. Using the formula of 80 and putting

$$(m_1+m_1)=-m_3 \text{ and } \frac{m_1\rho_1+m_2\rho_2}{m_1+m_2}=\rho_3,$$

we see that $m_1p\rho_1+m_2p\rho_2+m_3p\rho_3=0$, and $m_1+m_2+m_3=0$, *are the conditions that the three points $p\rho_1$, $p\rho_2$, $p\rho_3$ shall be collinear and the the vectors ρ_1, ρ_2, ρ_3, coplanar.*

84. For space of three dimensions we have (82)

$$m_1p\rho_1+m_2p\rho_2+m_3p\rho_3+m_4p\rho_4=0, \text{ and } m_1+m_2+m_3+m_4=0,$$

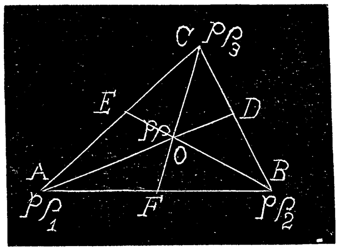
for the conditions that the four points $p\rho_1$, $p\rho_2$, $p\rho_3$, $p\rho_4$ shall be coplanar.

85. *From the equations of* 80, 83, 84 *we see that two points are independent, three or more collinear points are dependent* (10), *three non-collinear points are independent, four or more coplaner points are dependent, four non-coplaner points are independent, and any five or more points are dependent in solid space.*

86. The calculus of this chapter is evidently adapted to dealing with theorems concerning the collinearity of points in geometry, and the center of parallel forces in mechanics.

87. We conclude this chapter with an example. *Required to find whether the three medians of a triangle meet in a point.*

Let the vertices of a triangle ABC be located by the unit points $p\rho_1$, $p\rho_2$, $p\rho_3$, and D, E, F be the mid-points of the sides. We have then



$$D=\frac{p\rho_2+p\rho_3}{2},\quad E=\frac{p\rho_1+p\rho_3}{2}.$$

If $p\rho$ denote a unit point at O the intersection of AD and BE, and x, y, x', and y' arbitrary scalars, we may write

$$p\rho=xp\rho_1+y\frac{p\rho_2+p\rho_3}{2}=x'p\rho_2+y'\frac{p\rho_1+p\rho_3}{2}.$$

Then (20), $x=\frac{1}{2}y'$, $y=y'$; whence $x=\frac{1}{2}y$. But $x+y=1$ (77). Then $x=\frac{1}{3}$, $y=\frac{2}{3}$. Hence

$$p\rho=\tfrac{1}{3}p\rho_1+\tfrac{2}{3}\left(\frac{p\rho_2+p\rho_3}{2}\right)=\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3.$$

By symmetry we see that the intersection of AD and CF must be the same point. Or, supposing O to be the intersection of BE and CF, we may test A, O, and D for collinearity directly.

$$\begin{array}{lll} A & O & D \\ \tfrac{1}{3}p\rho_1 - 1 & (\tfrac{1}{3}p\rho_1+\tfrac{1}{3}p\rho_2+\tfrac{1}{3}p\rho_3) & +\tfrac{2}{3}(\tfrac{1}{2}(p\rho_2+p\rho_3))\equiv 0. \quad (83). \end{array}$$

It is evident that the above equations can be interpreted as equations of ordinary vector analysis by dropping the p's. In this way is shown the relation existing between point and vector analysis.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

127. **Proposed by P. S. BERG, A. M., Principal of Schools, Larimore, N. D.**

A man borrows \$1000 of a Building and Loan Association, and at the same time subscribes for 10 \$100-shares of stock. A membership fee of \$1 per share is charged. At the beginning of each month an installment of \$1 per share is paid, also 5% interest and 5% premium on the \$1000. The stock matures in 75 months and the debt is cancelled. What rate of interest does he pay per annum?

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

\$10 membership fee, \$10 installment, and \$8.33$\frac{1}{3}$ interest and premium per month=\$28.33$\frac{1}{3}$ amount paid down.

\$1000−\$28.33$\frac{1}{3}$=\$971.66$\frac{2}{3}$ actual amount received after deducting amounts paid at time of borrowing. On this amount, \$18.33$\frac{1}{3}$ per month is paid for 74 months.

$\therefore\ 18.33\frac{1}{3}=\dfrac{971.66\frac{2}{3}r(1+r)^{74}}{(1+r)^{74}-1}$, or $18\frac{1}{3}(1+r)^{74}-18\frac{1}{3}=971\frac{2}{3}r(1+r)^{74}$.

$(1+r)^{74}-1=53r(1+r)^{74}$.

$\therefore\ (1-53r)(1+r)^{74}=1$.

$\therefore\ \log(1-53r)+74\log(1+r)=0$.

$\therefore\ r=.0137$, and $12r=.1644=16.44\%$ per annum.

128. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

At what time is the figure 7, on the face of a clock, midway between the hour and minute hands?

Solution by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.; D. G. DORRANCE, Jr., Camden, N. Y.; G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; and the PROPOSER.

Put $7=a=$the given figure on face of clock.

Let $x=$distance the hour hand travels after a o'clock.

Then $5a-x=$distance the minute hand travels to fulfill the condition between a and $a+1$ o'clock.

$\therefore$ As the minute hand goes 12 times as fast as the hour hand, $5a-x=12x$, and $x=\dfrac{5a}{13}$.

This is the *first* position of the hour hand after a o'clock. For, it will be observed, there are thirteen different positions in all: one after each hour and one at $2a$ o'clock.

For the *second* position, which is between $a+1$ and $a+2$ o'clock, $60+5a-x=$distance the minute hand has to travel. Whence, $60+5a-x=12x$, and $x=\dfrac{60+5a}{13}$.

Now let n represent these 13 positions of the hour hand. Then $x=\dfrac{60(n-1)+5a}{13}=$the nth position of the hour hand after a o'clock.

The time of day for the different positions is, *before* $2a$ o'clock, $5a-\dfrac{60(n-1)+5a}{13}$, or $\dfrac{60(a-n+1)}{13}$ minutes *past* $a+n-1$ o'clock; and, *after* $2a$ o'clock, $\dfrac{60(n-1)+5a}{13}-5a$, or $\dfrac{60(n-1-a)}{13}$ minutes *to* $a+n-1$ o'clock.

Substituting 7 for a, and 1, 2, 3....13, consecutively, for n, we obtain the following thirteen times of day when 7 is midway between the hour and minute hands: $32\frac{4}{13}$ minutes past 7, $27\frac{9}{13}$ minutes past 8, $23\frac{1}{13}$ minutes past 9,

$18\frac{6}{13}$ minutes past 10, $13\frac{11}{13}$ minutes past 11, $9\frac{3}{13}$ minutes past 12, $4\frac{8}{13}$ minutes past 1, 2 o'clock, $4\frac{8}{13}$ minutes to 3, $9\frac{3}{13}$ minutes to 4, $13\frac{11}{13}$ minutes to 5, $18\frac{6}{13}$ minutes to 6, and $23\frac{1}{13}$ minutes to 7.

Also solved by *JOSIAH H. DRUMMOND*, *P. S. BERG*, *ELMER SCHUYLER*, *H. C. WHITAKER*, and *J. SCHEFFER*.

129. Proposed by J. W. DAPPERT, Civil Engineer and Surveyor, Taylorville, Ill.

"A Minion, agile, in stature small
Panting came to great Diana's Hall,
Bearing a marble globe upon his shoulders,
Measuring one inch in its diameters.
He rolled it to the northeast corner of the Hall,
Left touching the northern and eastern walls;
Then following came three demi-gods in white,
Each bearing a globe of lustrous metal bright;
One of iron, copper one, and one of silver;
And they placed them in the order given,
Touching each the other, and at the same time,
Touching each the side walls, in a direct line,
The iron touching the marble, and its other side
Resting against the silver, in its glory and pride,—
All resting upon the oaken floor; and then
With heavy tread, and puff, and roar, Atlas came
Bearing a huge golden sphere, that filled the Hall,
Touching the four sides, floor and ceiling, and all
Radiant with beauty, resting against the silvery ball,
Making the globe's diameters in the room diagonal."

"Tell me, all ye who mathematics know:
What size the copper sphere, and oh!
How large the iron globe? How great
The golden globe; immaculate?
The silver sphere, how great? What size?
And if presented as a prize,
What value do you hold
Would be the sphere of gold?"

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

A cube, in form, is great Diana's Hall:
The massive sphere of gold, admired by all,
With ceiling, sides and floor in contact is.
Within the northeast corner's boundaries
Are found the other spheres, in number four,
Each tangent to the corner's sides and floor;
Besides, the sizes of the sphere are such
That each one with its neighbor is in touch.
The order, as they to the corner rolled,
Was marble, iron, copper, silver, gold.
The hall's dimensions, length and breadth and height,
And golden sphere's diameter are quite
The same in measurement; let this be b.
The hall's diagonal, which we'll name c,
Is quickly found *b times square root of 3*. $[b\sqrt{3}]$
The points of contact of the spheres we find
Are in the hall's diagonal confined;
So, too, the centers of the spheres are there.
If hall with golden sphere we now compare,
Outside the sphere are equal ends of c;
And, known as d, each end is found to be
Square root of 3 less one times half of b. $[\frac{1}{2}b(\sqrt{3}-1)]$
The ratio of diameters to find
Of tangent spheres, must next be borne in mind.
Take a as silver sphere's diameter;

Then d is found to equal, we aver,
Square root of 3 plus one times half of a. $\lfloor\frac{1}{2}a(\sqrt{3}+1)\rfloor$
Equating d's two different values, weigh
Results, and see *square root of 3 plus 2*. $(\sqrt{3}+2\rfloor$
Of b to a the ratio, come to view.
This ratio known as t; then find it true
That of diameters of any two
Contiguous spheres within the corner's space
The ratio is the self-same t, a case
Of beauty mathematical. We now
From demonstrations rest the knitted brow,
And the diameters of all the spheres
Announce. Within the problem it appears
That the diameter of marble sphere
Is just *one inch*. From ratio t 't is clear
That iron sphere's diameter is t;
Then that of copper sphere *t squared* must be;
Whence that of silver sphere is *cube of t*;
And the diameter of golden sphere
As *t involved to fourth power* must appear.
Three hundred seventy-three as hundredths is
The value of the ratio t; and this $[t=3.73+]$
To powers second, third and fourth involved,
Yields as results, approximately solved, $\lfloor 13.92+]$
Nine tenths to thirteen added, fifty-two, $\lfloor 51.98+]$
And ninety-seven multiplied by two. $[193.99+]$

The golden sphere is wondrous as to size;
And in regard to value as a prize,
Not all our country's golden output coined
In her existing years, together joined,
Could purchase its great worth. In numbers round,
Eight hundred four of millions will be found
The dollars that in purest gold abound
In this great sphere. Besides it would confound
The efforts of the mind that should aspire
To measure the extent of thinnest wire
To which there might be drawn this golden mass.
From Earth to Sun this slender thread could pass,
Return again to Earth, and eight times more
The circuit make, and still have end galore
To stretch from Earth to Moon strands eighteen score.

ALGEBRA.

104. Prize Problem. $2.50 for the best solution.

Compute to three decimal places each of the roots of the equation $x^2+y=2$, $x+y^2=6$.

I. Solution by AGNES E. SCHEFFER, Hagerstown, Md.

From the first of these equations we have $y=2-x^2$, and substituting this in the second, we have $x^4-4x^2+x-2=0$.

Factoring we have $x^2(x^2-4)+(x-2)=0$, or $(x-2)(x^3+2x^2+1)=0$.

$\therefore$ $x-2=0$, and $x^3+2x^2+1=0$.

From the former we obtain $x=2$ and then its simultaneous value $y=-2$, obtained from the first of the original equations.

By solving the cubic equation $x^3+2x^2+1=0$, we obtain three more roots for x.

Putting $x=z-\frac{2}{3}$, we get $z^3-\frac{4}{3}z+\frac{43}{27}=0$, and employing Cardon's formula we have

$$y=\sqrt[3]{[-\tfrac{43}{27}+\tfrac{1}{54}\sqrt{(1593)}]}+\sqrt[3]{[-\tfrac{43}{54}-\tfrac{1}{54}\sqrt{(1593)}]},\text{ or}$$

$$z=\tfrac{1}{3}\{\sqrt[3]{[-\tfrac{43}{2}+\tfrac{1}{2}\sqrt{(1593)}]}+\sqrt[3]{[-\tfrac{43}{2}-\tfrac{1}{2}\sqrt{(1593)}]}\},\text{ and}$$

$$x=-\tfrac{2}{3}+\tfrac{1}{3}\{\sqrt[3]{[-\tfrac{43}{2}+\tfrac{1}{2}\sqrt{(1593)}]}+\sqrt[3]{[-\tfrac{43}{2}-\tfrac{1}{2}\sqrt{(1593)}]}\}=-2.205569.$$

Dividing x^3+2x^2+1 by $x+2.205569$ we get $x^2-205569x+.4533966$, and now the equation $x^2-.205569x+.4533966=0$ furnishes us the other two roots of x, viz: $x=.102784\pm.665456\sqrt{-1}$.

The simultaneous values of y we obtain from the equation $x^2+y=2$, viz: $y=2-x^2$.

Thus, we find the following four sets of the simultaneous values of x and y:

$x=2$	$x=-2.205569$	$x=.102784\pm 665456\sqrt{-1}$
$y=-2$	$y=-2.864534$	$y=2\ 432267\mp.136796\sqrt{-1}$

II. Solution by J. W. YOUNG, Fellow and Assistant in Mathemetics, Ohio State University, Columbus, O.

I. Solve (1) for y, substitute in (2) and obtain

$$x^4-4x^2+x-2=0.$$

In the application of Horner's method or by inspection we see that $2=x_1$ is a root. By dividing out this root we obtain for the equation giving the remaining roots

$$x^3+2x^2+1=0\ldots.(3).$$

This equation has a pair of complex roots. Denoting the real root by x_2, and applying Horner's process, we find

$$x_2=-2.2055+.$$

Denoting the other roots x_3, x_4 by $\alpha\pm i\beta$ and observing that the sum of the roots of (3) equals -2, we have

$$2\alpha-2.2055=-2.$$

Whence $\alpha=0.1027+$.

Similarly, the product of the roots,

$$x_2(\alpha^2+\beta^2)=-1.$$

Whence $\beta=0.6654+$.

Hence the complex roots are x_3, $x_4=0.1027\pm i0.6654+$.

Collecting results and calculating the corresponding values of y, we have, as a complete solution of the original system.

$$x_1=2. \qquad y_1=-2.$$

$$x_2=-2.2055 \qquad y_2=-2.8642$$

$$\left.\begin{matrix}x_3\\x_4\end{matrix}\right\}=0.1027\pm i0.6654 \qquad \left.\begin{matrix}y_3\\y_4\end{matrix}\right\}=2.4323\mp i0.1367$$

II. Since the cubic giving the incommensurable roots has a pair of complex roots, Cardan's solution may be applied. In the cubic (3) put $x=z-\frac{2}{3}$, and obtain

$$z^3-\tfrac{12}{9}z+\tfrac{43}{27}=0\ldots.(4).$$

Put $z=\sqrt[3]{p}+\sqrt[3]{q}$, and cube. Then

$$z^3-3\sqrt[3]{p}\sqrt[3]{q}z-(p+q)=0\ldots.(5).$$

Comparing coefficients in (4) and (5), we have

$$p+q=-\tfrac{43}{27},\ pq=\tfrac{64}{729}.$$

Whence $p=-0.0572$ and $\sqrt[3]{p}=-0.3853$, $q=-1.5354$ and $\sqrt[3]{q}=-1.1536$.

Hence the real root of $(4)=\sqrt[3]{p}+\sqrt[3]{q}=-1.5389$ and $x_2=z-\frac{2}{3}=-1.5389-0.7666=-2.2055$ as before.

The complex roots are given by $w\sqrt[3]{p}+w^2\sqrt[3]{q}$ and $w^2\sqrt[3]{p}+w\sqrt[3]{q}$, where $w=\frac{1}{2}(-1\pm i\sqrt{3})$. Computing these roots by the above formulae, we obtain as before

$$x_3,\ x_4=0.1027\pm i0.6654.$$

III. In equation (3) substitute $x=\alpha+i\beta$, and obtain, after separating real and imaginary parts

$$\alpha^3+2\alpha^2+1-\beta^2(\beta\alpha+2)=0\ldots.(6),$$

$$\beta(\beta^2-3\alpha^2-4\alpha)=0\ldots.(7).$$

From (7), $\beta=0$,

$$\beta^2=3\alpha^2-4\alpha\ldots.(8).$$

For $\beta=0$ (x real) we have then $\alpha^3+2\alpha^2+1=0$ (from (6)) giving the real root, $x_2=-2.2055$.

For $\beta^2=3\alpha^2-4\alpha$, we have from (6),

$$8\alpha^3+16\alpha^2+8\alpha-1=0.$$

Solving this we obtain $\alpha=0.1027$ as before. And from (8) $\beta=0.6654$.

IV. We may write (1) and (2) in the form

$$x^2-4=-(2+y)$$

$$x-2=4-y^2.$$

From which immediately $x=2$, $y=-2$.

Dividing we have $x+2=-\dfrac{1}{2-y}$, or $y=\dfrac{2x+5}{x+2}$.

Substitute in (1), and obtain $x^3+2x^2+1=0$. Solve this by any of the above methods.

V. Add (1) and (2) and complete squares. Then

$$x^2+x+\tfrac{1}{4}+y^2+y+\tfrac{1}{4}=8+\tfrac{2}{4} \text{ or } (x+\tfrac{1}{2})^2+(y+\tfrac{1}{2})^2=\tfrac{25}{4}+\tfrac{9}{4}.$$

Whence, from the four possible corresponding values of x and y, we may pick out one set which satisfies the original system, namely $x=2$, $y=-2$.

Also solved by *GEO. R. BERRY.*

NOTE. The donor of this prize has acted as judge on the merits of the several solutions, and his decision is that the two published solutions are of equal merit. In accordance with this decision, the prize money has been equally divided between Miss Scheffer and Mr. Young. We might say that there has been only one solution sent in to the prize problem in Mechanics. This solution is defective. The problem is, therefore, open to all our contributors for solution. EDITOR F.

105. Proposed by CHARLES E. MYERS, Canton, O.

Solve for x the following: $a^{\log(a^{x^2})}=m^{\log(m)}$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; J. K. ELLWOOD, A. M., Colfax School, Pittsburg, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; COOPER D. SCHMITT, A. M., University of Tennessee, Knoxville, Tenn.; W. F. SHAW, Austin, Tex.; and ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$a^{\log(a^{x^2})}=m^{\log m}$ may be written $a^{x^2\log a}=m^{\log m}$.

$\therefore\ x^2(\log a)^2=(\log m)^2$.

$\therefore\ x=\pm(\log m/\log a)$.

GEOMETRY.

130. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics in Drury College, Springfield, Mo.

If the points x, y, z divide the strokes $c-b$, $a-c$, $b-a$, in the same ratio r, and the triangles x, y, z and a, b, c are similar, either $r=1$ or both triangles are equilateral. [From Harkness and Morley's *Introduction to the Theory of Functions*, page 26].

Solution by the PROPOSER.

Let x, y and z denote the points, dividing $c-b$, $a-c$, and $b-a$, respectively, in the given ratio r.

Then $x=\dfrac{c+br}{1+r}$, $y=\dfrac{a+cr}{1+r}$, and $z=\dfrac{b+ar}{1+r}$.

The condition that a, b, c, and x, y, z form similar triangles is

$$\begin{vmatrix} a & x & 1 \\ b & y & 1 \\ c & z & 1 \end{vmatrix}=0\ldots.(1).$$

Substituting in this equation, the values of x, y, and z, and reducing, we obtain

$$r(bc+ac+ab-a^2-b^2-c^2)=bc+ac+ab-a^2-b^2-c^2.$$

From this equation we find that $r=1$, unless

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=0\ldots.(2).$$

But (2) is the condition that the triangle a, b, c is equilateral. Therefore, either $r=1$, or else both triangles are equilateral.

Also solved by *G. B. M. ZERR*, *J. W. YOUNG*, and *FRANK A. GRIFFIN*.

131. Proposed by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.

Prove that $\lambda+\mu\omega+\nu\omega^2$, where λ, μ, ν are integers whose sum is ± 1, represents the points of a quilt formed by regular hexagons. ω=primitive cube root of unity. [From Harkness and Morley's *Introduction to Theory of Functions.*]

Solution by the PROPOSER.

$$\omega=-\tfrac{1}{2}+i(\tfrac{1}{2}\sqrt{3}),\ \omega=-\tfrac{1}{2}-i(\tfrac{1}{2}\sqrt{3})\ (i^2=-1).$$

Then $\lambda+\mu\omega+\nu\omega^2=\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu+i(\mu-\nu)\sin\frac{1}{3}\pi$. Taking rectangular coördinates this quantity represents the points (x, y), when

$$\left.\begin{array}{l} x=\lambda-\frac{1}{2}\mu-\frac{1}{2}\nu \\ y=(\mu-\nu)\sin\frac{1}{3}\pi \end{array}\right\} \begin{array}{l} \lambda+\mu+\nu=\pm 1 \\ (\lambda,\ \mu,\ \nu \text{ are integers}) \end{array}$$

$y=n\sin\frac{1}{3}\pi$, when $n=(\mu-\nu)$=any integer.
Then $\mu=n+\nu$.
Substituting in x and in $\lambda+\mu+\nu=\pm 1$, we obtain

$$\left.\begin{array}{r} 2\lambda-2\nu-n=2x \\ \lambda+2\nu+n=\pm 1 \end{array}\right\}\ldots.(1).$$

$\therefore x=\frac{1}{2}(3\lambda\pm 1)$.

The points required are, then, those whose coördinates are $[\frac{1}{2}(3\lambda\pm 1,\ n\sin\frac{1}{3}\pi]$, where λ, n are any integers with the one restriction that when n is *odd*, λ is even, and *vice versa*. This restriction is evident, since (1) shows that $\lambda+n$ must be *odd*.

We have then following values of x and y :

A.

For $n=0, 2, 4, 6$, etc.

$\lambda=$	1	3	5	7	9	etc.
$x=$	1 2	4 5	7 8	10 11	13 14	

For $n=1, 3, 5, 7$, etc.

$\lambda=$	0	2	4	6	8	
$x=$	$-\frac{1}{2}$ $\frac{1}{2}$	$\frac{5}{2}$ $\frac{7}{2}$	$\frac{11}{2}$ $\frac{13}{2}$	$\frac{17}{2}$ $\frac{19}{2}$	$\frac{23}{2}$ $\frac{25}{2}$	

$\Big\}\ y=n\sin\frac{1}{3}\pi$.

Similarly for negative values of n, λ.

MECHANICS.

96. Proposed by GEORGE R. DEAN, Professor of Mathematics. University of Missouri School of Mines and Metallurgy, Rolla, Mo.

Two particles, subject to their mutual attraction and that of a fixed center, move in a plane containing the center. Find the motion under the law of the inverse square.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Take the center of force as origin. Let m, m_1, m_2 be the masses of the center of force and particles, respectively. r, r_1, ρ the distances of the particles from the center of force and from each other, respectively. (x, y), (x', y') the coördinates of the particles. The differential equations of motion of the two particles relative to the center of force are

$$\left.\begin{aligned}\frac{d^2x}{dt^2} &= -\frac{m+m_1}{r^3}x+\frac{x'-x}{\rho^3}m_2-\frac{m_2x'}{r_1^3}\\ \frac{d^2y}{dt^2} &= -\frac{m+m_1}{r^3}y+\frac{y'-y}{\rho^3}m_2-\frac{m_2y'}{r_1^3}\end{aligned}\right\}\ldots\ldots(1).$$

$$\left.\begin{aligned}\frac{d^2x'}{dt^2} &= -\frac{m+m_2}{r_1^3}x'+\frac{x-x'}{\rho^3}m_1-\frac{m_1x}{r^3}\\ \frac{d^2y}{dt^2} &= -\frac{m+m_2}{r_1^3}y'+\frac{y-y'}{\rho^3}m_1-\frac{m_1y}{r^3}\end{aligned}\right\}\ldots(2).$$

Where $\rho=\sqrt{[(x'-x)^2+(y'-y)^2]}$.

Multiply (1) by $2m_1\dfrac{dx}{dt}-2m_1\dfrac{m_1\dfrac{dx}{dt}+m_2\dfrac{dx'}{dt}}{m+m_1+m_2}$.

$$2m_1\frac{dy}{dt}-2m_1\frac{m_1\dfrac{dy}{dt}+m_2\dfrac{dy'}{dt}}{m+m_1+m_2}.$$

Multiply (2) by $2m_2\dfrac{dx'}{dt}-2m_2\dfrac{m_2\dfrac{dx'}{dt}+m_1\dfrac{dx}{dt}}{m+m_1+m_2}$,

and $2m_2\dfrac{dy'}{dt}-2m_2\dfrac{m_2\dfrac{dy'}{dt}+m_1\dfrac{dy}{dt}}{m+m_1+m_2}$.

Adding the four products we get

$$2m_1\left(\frac{dx}{dt}.\frac{d^2x}{dt^2}+\frac{dy}{dt}.\frac{d^2y}{dt^2}\right)+2m_2\left(\frac{dx'}{dt}.\frac{d^2x'}{dt^2}+\frac{dy'}{dt}.\frac{d^2y'}{dt^2}\right)$$

$$-2\frac{m_1\dfrac{dx}{dt}+m_2\dfrac{dx'}{dt}}{m+m_1+m_2}\left(m_1\frac{d^2x}{dt^2}+m_2\frac{d^2x'}{dt^2}\right)$$

$$-2\frac{m_1\dfrac{dy}{dt}+m_2\dfrac{dy'}{dt}}{m+m_1+m_2}\left(m_1\frac{d^2y}{dt^2}+m_2\frac{d^2y'}{dt^2}\right)$$

$$-2m\left(\frac{m_1}{r^2}\frac{dr}{dt}+\frac{m_2}{r_1^2}\frac{dr_1}{dt}\right)-2\frac{d}{dt}\left(\frac{m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}\right)=0.$$

Integrating we get

$$m_1[(dx/dt)^2+(dy/dt)^2]+m_2[(dx'/dt)^2+(dy'/dt)^2]$$

$$-\frac{[(m_1dx/dt+m_2dx'/dt)^2+(m_1dy/dt+m_2dy'/dt)^2]}{m+m_1+m_2}$$

$$-2m\left(\frac{m_1}{r}+\frac{m_2}{r_1}\right)-\frac{2m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}=A,$$

or $mm_1[(dx/dt)^2+(dy/dt)^2]+mm_2[(dx'/dt)^2+(dy'/dt)^2]$

$$+m_1m_2[(dx'/dt-dxdt)^2+(dy'/dt-dy/dt)^2]$$

$$-2(m+m_1+m_2)\left[m\left(\frac{m_1}{r}+\frac{m_2}{r_1}\right)+\frac{m_1m_2}{\sqrt{[(x'-x)^2+(y'-y)^2]}}\right]=A.$$

The *vis viva* equation of motion.

DIOPHANTINE ANALYSIS.

82. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

In the series $1^3+3^3+5^3$.... find n so that the nth term and the sum of n terms shall both be squares.

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.

The conditions of the problem require $1^3+3^3+5^3+....+(2n-1)^3 = n^2(2n^2-1)=\square$, and $(2n-1)^3=\square$.

Whence it follows that $2n-1=\square$, and $2n^2-1=\square$.

$2n-1=\square$, when $n=r^2+(r-1)^2$, as 1, 5, 13, 25, 41, etc.

Hence $2n^2-1=2[r^2+(r-1)^2]^2-1=\square$.

Whence $2[r^2+(r-1)^2]^2=\square+1$, two times a square equals the sum of two squares, the general formula for which is $2(p^2+q^2)^2=[(p+q)^2-2q^2]^2+[(p-q)^2-2q^2]^2$.

Then $p^2+q^2=r^2+(r-1)^2$....(1), and $(p-q)^2-2q^2=\pm1$....(2).

From (1), put $p=r$ and $q=(r-1)$; and substituting in (2) we obtain $1-2(r-1)^2=\pm1$, or $2(r-1)^2=0$ or 2.

Whence $r-1=0$ or ±1, and $r=1$ or 2 or 0.

Substituting these values of r in $n=r^2+(r-1)^2$, we find $n=1$ and 5, apparently the only integral values.

When $n=1$, $2n-1=1$, $n^2(2n^2-1)=1$, and $(2n-1)^3=1$.

When $n=5$, $2n-1=9=3^2$, $n^2(2n^2-1)=1^3+3^3+5^3+7^3+9^3=35^2$, and $(2n-1)^3=(3^2)^3=(3^3)^2=27^2$.

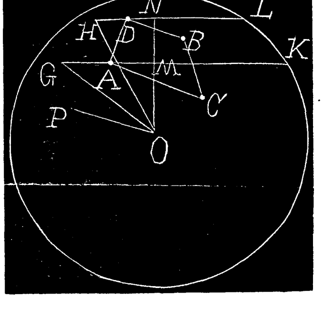
Also solved by *G. B. M. ZERR*, *J. SCHEFFER*, *COOPER D. SCHMITT*, and the *PROPOSER*.

AVERAGE AND PROBABILITY.

88. Proposed by G. B M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

Find the average volume of the tetrahedron formed by joining four random points in a sphere.

Solution by the PROPOSER.



Let GK be the diameter of the section of the sphere made by a plane through the three random points A, B, C; HL the diameter of a parallel section through the fourth random point D; M, N the centers of these sections, respectively; O the center of the sphere; OP a line such that AB is always parallel to the plane MOP.

Let $OG=OH=r$, $MA=x$, $AB=y$, $AC=z$, $DN=u$, $\angle GOM=\theta$, $\angle BAM=\varphi$, $\angle CAM=\psi$, $\angle HON=\beta$, $\angle MOP=\lambda$, and the angle the plane POM makes with a fixed plane through $OP=\rho$.

An element of the sphere at A is $r\sin\theta d\theta 2\pi x dx$; at B, $y^2 dy d\varphi d\lambda$; at C, $\sin(\varphi+\psi)\sin\lambda z^2 dz d\psi d\rho$; at D, $r\sin\beta d\beta 2\pi u du$.

The limits of θ are 0 and π; of β, 0 and θ; of x, 0 and $r\sin\theta=x'$ and tripled; of u, 0 and $r\sin\beta=u'$ and sextupled; of φ, $-\frac{1}{2}\pi$ and $+\frac{1}{2}\pi$; of ψ, $-\varphi$ and $\frac{1}{2}\pi$; of λ, 0 and π; of ρ, 0 and π; of y, 0 and $2x\cos\varphi=y'$; of z, 0 and $2x\cos\psi=z'$.

The area of the triangle $ABC=\frac{1}{2}yz\sin(\varphi+\psi)$.

Altitude of tetrahedron$=r(\cos\beta-\cos\theta)$.

Volume of tetrahedron$=\frac{1}{6}ryz\sin(\varphi+\psi)(\cos\beta-\cos\theta)$.

Since the whole number of ways four points can be taken is $(\frac{4}{3}\pi r^3)^4$, the required average triangle is

$$\triangle=\frac{729}{128\pi^4 r^{12}}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\int_0^{y'}\int_0^{z'}\tfrac{1}{6}ryz\sin(\varphi+\psi)$$

$$\times(\cos\beta-\cos\theta)r\sin\theta d\theta 2\pi x dx r\sin\beta d\beta 2\pi u du \sin(\varphi+\psi)d\varphi d\psi \sin\lambda d\lambda d\rho y^2 dy z^2 dz$$

$$=\frac{243}{16\pi^2 r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\int_0^{y'}\sin\theta x^5\sin\beta u\sin^2(\varphi+\psi)\cos^4\psi$$

$$\times\sin\lambda(\cos\beta-\cos\theta)d\theta d\beta dx du d\varphi d\psi d\lambda d\rho y^3 dy$$

$$=\frac{243}{4\pi^2 r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\int_0^\pi\int_0^\pi\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2$$

$$\times(\varphi+\psi)\cos^4\varphi\cos^4\psi\sin\lambda d\theta d\beta dx du d\varphi d\psi d\lambda d\rho$$

$$=\frac{243}{2\pi r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\int_{-\Phi}^{\frac{1}{2}\pi}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u\sin^2(\varphi+\psi)$$

$$\times\cos^4\varphi\cos^4\psi d\theta d\beta dx du d\varphi d\psi$$

$$=\frac{81}{64\pi r^9}\int_0^\pi\int_0^\theta\int_0^{x'}\int_0^{u'}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u(15\pi\cos^4\varphi$$

$$-12\pi\cos^6\varphi+30\varphi\cos^4\varphi-24\varphi\cos^6\varphi-4\sin\varphi\cos^7\varphi+30\sin\varphi\cos^5\varphi)d\theta d\beta dx du d\varphi$$

$$=\frac{1215\pi}{256r^9}\int_0^\pi\int_0^\theta\int_0^{x}\int_0^{u'}\sin\theta\sin\beta(\cos\beta-\cos\theta)x^9 u d\theta d\beta dx du$$

$$=\frac{1215\pi}{512r^7}\int_0^\pi\int_0^\theta\int_0^{x'}\sin\theta\sin^3\beta(\cos\beta-\cos\theta)x^9 d\theta d\beta dx$$

$$=\frac{243\pi r^3}{1024}\int_0^\pi\int_0^\theta\sin^{11}\theta\sin^3\beta(\cos\beta-\cos\theta)d\theta d\beta$$

$$=\frac{81\pi r^3}{4096}\int_0^\pi\sin^{11}\theta(8-4\sin^2\theta-\sin^4\theta-8\cos\theta)d\theta=\frac{36\pi r^3}{715}.$$

MISCELLANEOUS.

81. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.

A cask in the form of a middle frustum of a spheroid, middle diameter $2b$, end diameters each $2c$, length $2d$, is lying in a horizontal position. The distance from middle of top to water is $b+e$, $e<b-c$. How much water is in the cask?

Solution by the PROPOSER.

Let $x^2/a^2+(y^2+z^2)/b^2=1$, be the equation to the spheroid.
Then $a^2=b^2d^2/(b^2-c^2)$, $z=(1/a)\sqrt{[b^2(a^2-x^2)-a^2y^2]}$.

$$\therefore\ V=(4/a)\int_0^d\int_e^{(b/a)\sqrt{(a^2-x^2)}}\sqrt{[b^2(a^2-x^2)-a^2y^2]}dxdy$$

$$=\frac{\pi b^2}{a^2}\int_0^d(a^2-x^2)dx-(2e/a)\int_0^d\sqrt{[a^2(b^2-e^2)-b^2x^2}dx$$

$$-\frac{2b^2}{a^2}\int_0^d(a^2-x^2)\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-x^2)}}\right)dx=\frac{\pi b^2}{3a^2}(3a^2d-d^3)$$

$$-\frac{de}{a}\sqrt{[a^2(b^2-e^2)-b^2d^2]}-\frac{ae(b^2-e^2)}{b}\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right)$$

$$-\frac{2b^2d}{3a^2}(3a^2-d^2)\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-d^2)}}\right)+\frac{2b^2e}{3a}\int_0^d\frac{x^2(3a^2-x^2)dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}$$

$$D=\frac{2b^2e}{3a}\int_0^d\frac{x^2(3a^2-x^2)dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}=\frac{2b^2e}{3a}\int_0^d\frac{x^2dx}{\sqrt{[a^2(b^2-e^2)-b^2x^2}}$$

$$+\frac{4ab^2e}{3}\int_0^d\frac{x^2dx}{(a^2-x^2)\sqrt{[a^2(b^2-e^2)-b^2x^2}}.$$

Let $bx=a\sqrt{(b^2-e^2)}\sin\theta$, $\theta_1=\sin^{-1}\{bd/[a\sqrt{(b^2-e^2)}]\}$.

$$\therefore\ D=\frac{2ae(b^2-e^2)}{3b}\int_0^{\theta_1}\sin^2\theta d\theta+\frac{4abe(b^2-e^2)}{3}\int_0^{\theta_1}\frac{\sin^2\theta d\theta}{b^2-(b^2-e^2)\sin^2\theta}$$

$$=\frac{4ab^2}{3}\tan^{-1}\left(\frac{de}{\sqrt{[a^2(b^2-e^2)-b^2d^2]}}\right)-\frac{ae(e^2+3b^2)}{3b}\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right)$$

$$-(de/3a)\sqrt{[a^2(b^2-e^2)-b^2d^2]}.$$

$$\therefore\ V=\frac{b^2d}{3a^2}(3a^2-d^2)\left[\pi-2\sin^{-1}\left(\frac{ae}{b\sqrt{(a^2-d^2)}}\right)\right]-\frac{4de}{3a}\sqrt{[a^2(b^2-e^2)-b^2d^2]}$$

$$+\frac{4ab^2}{3}\tan^{-1}\left(\frac{de}{\sqrt{[a^2(b^2-e^2)-b^2d^2]}}\right)+\frac{2ae}{3b}(e^2-3b^2)\sin^{-1}\left(\frac{bd}{a\sqrt{(b^2-e^2)}}\right).$$

But $a=bd/\sqrt{(b^2-c^2)}$.

$$\therefore\ V=\frac{d(2b^2+c^2)}{3}[\pi-2\sin^{-1}(e/c)]-(4de/3)\sqrt{(c^2-e^2)}$$

$$+\frac{2d}{3\sqrt{(b^2-c^2)}}\left\{2b^3\sin^{-1}\left[e/c\sqrt{\left(\frac{b^2-c^2}{b^2-e^2}\right)}\right]+e(e^2-3b^2)\sin^{-1}\sqrt{\left(\frac{b^2-c^2}{b^2-e^2}\right)}\right\}.$$

A good solution, with diagram, was received from *J. SCHEFFER*.

BIOGRAPHICAL SKETCH OF SYLVESTER ·ROBINS.

Sylvester Robins was born in Union Township, Unterdon County, New Jersey, December 14, 1834, and died at his home in North Branch, New Jersey, April 25, 1900, in the 66th year of his age.

During early boyhood, he attended the common school in the neighborhood, and very early manifested a deep interest in his studies.

At the age of 12 he entered the preparatory school of Rev. John Derveer, D. D., at Easton, Pennsylvania, where he continued for six years, during the last two years of which he was employed as assistant teacher. During the years from 1852 to 1873, Mr. Robins was engaged as teacher at Cedar Grove, Readington, and Bloomsberg, New Jersey, and for six and a half years at Easton, Pennsylvania.

In 1858 Mr. Robins was married to Miss Sarah J. Bird, and of this marriage there were born seven children—one of whom is Edward R. Robins, Instructor in Mathematics in Albany Academy, Albany, New York, and who is the author of *Algebra Reviews*, a small book published by Ginn & Co. before its author had reached the age of 21.

Mr. Robins loved mathematics and lamented the fact that his early opportunities had been so limited, and delighted to revel in finding series of rational triangles, rational trapezoids, or rational parallelopipeds, for which he found "keys." Numerous problems of this character are proposed by him in the MONTHLY, and in the *Mathematical Magazine* and *Mathematical Visitor* published by Dr. Artimas Martin. It was somewhat surprising, how, in continued fractions from different numbers he would see the "key" to some kind of a mathematical figure. In his last letter to Mr. C. A. Roberts, an intimate friend of his and a mathematician interested in very much the same line of investigation, Mr. Robins wrote, "I hope you will indulge the spirit in which I write and pardon the writer, who can no more help his desire to ask you for aid, than he can escape the sight of triangles in the leaves, or of parallelopipeds in the bricks under his feet."

Mr. Robins was a contributor to the MONTHLY from the first, and there are yet a number of his contributions in his favorite line of work in the hands of the editor.

BOOKS.

Plane and Solid Geometry.—Inductive Method. By A. A. Dodd, M. S. D., S. B., and B. F. Chace, Teachers of Mathematics in the Manual Training High School, Kansas City, Mo. 8vo. Cloth, 406 pages. Price, $1.00. Kansas City: Hudson-Kimberly Publishing Co.

This work presupposes, on the part of the student, a knowledge of Inventional or Constructional Geometry. The purpose of the work is good, and in the hands of a skillful and well trained teacher the results from the study of such a book would be very good; but in the hands of a teacher whose knowledge of geometry is somewhat deficient, its study would certainly be unsatisfactory. The authors have the right view of presenting the subject, and are to be congratulated in the courage they have manifested in presenting the work for public recognition. B. F. F.

A Brief History of Mathematics. An Authorized Translation of Dr. Karl Fink's Geschichte der Elementar-Mathematik. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Principal of the State Normal School at Brockport, New York. 8vo, Red Cloth, 333 pages. Price $1.50. Chicago: The Open Court Publishing Co.

This work briefly states the facts of mathematical history. It is not a book of anecdotes, nor one of biography. The author systematically traces the development of the science of mathematics from the earliest times down to the present. Geometry is reviewed from the primitive ideas of the Babylonians to the projective and differential geometry and the science of n-dimensional space and trigonometry from the ideas of Ahmes to the refined notions of recent times. B. F. F.

Elements of Algebra. By Wooster Woodruff Beman, Professor of Mathematics in the University of Michigan, and David Eugene Smith, Principal of the State Normal School at Brockport, New York.

In this text-book, the authors have followed their usual plan of allowing the light of modern mathematics to shine in upon the old. "And this is the condemnation, that light has come into the world and men love darkness rather than light." An examination of this book will commend it to all good teachers of algebra. B. F. F.

Bestimmung der Coefficienten welche bei der Berechnung der Intergale

$$\int\frac{x^n dx}{\sqrt{1+ax+bx^2}} \text{ und } \int\frac{x^n dx}{\sqrt{1+ax^2+bx^2+cx^3}}$$

auftreten. Von Henry Benner. Pamphlet, 60 pages. Chicago: Ginn & Co.

This dissertation was offered by its author as a partial fulfillment of the requirements for the degree of Doctor of Philosophy at the University at Erlangen, Germany.

B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the June-July Number.]

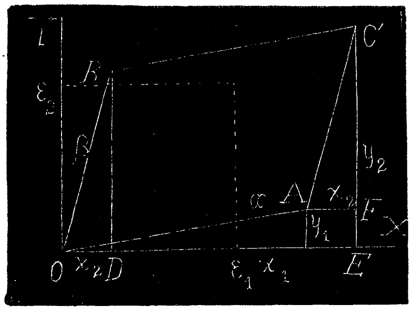
CHAPTER VII.

GEOMETRICAL MULTIPLICATION.

88. Basing our investigation on the fundamental law of combinatory multiplication (34), *let us seek the product of a non-posited point* (76) *and two vectors.* The vectors are thought of as denoting merely translation a given distance in a given direction (See 4—9). Let p denote the point and α and β the vectors.

Suppose

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2$$
$$\beta = x_2\varepsilon_1 + y_2\varepsilon_2$$



where ε_1 and ε_2 are unit vectors at right angles to each other. Then by 45

$$[p\alpha\beta] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} [p\varepsilon_1\varepsilon_2].$$

Now the determinant $x_1y_2 - x_2y_1$ is the difference between two rectangles.

Let us seek the relation, if any exists, between this area and that of the parallelogram $AOCB$. We have

$$OACB = OBCE - OACE = BCED - OAFE$$
$$= \tfrac{1}{2}(y_1 + 2y_2)x_1 - \tfrac{1}{2}(x_1 + 2x_2)y_1 = x_1 y_2 - x_2 y_1.$$

The equation $[p\alpha\beta] = (x_1 y_2 - x_2 y_1)[p\varepsilon_1 \varepsilon_2]$ shows, therefore, that if $p\varepsilon_1\varepsilon_2$ is taken to denote the area of the unit square the two sides of which are ε_1 and ε_2, $p\alpha\beta$ denotes the area of the parallelogram whose adjacent sides are α and β.

Now to assume that $p\varepsilon_1\varepsilon_2$ is the area of the square is a perfectly natural assumption analogous to the theorem in geometry which says, The area of a rectangle equals the product of the base by the altitude. *Thus Grassman was led to define the product $\alpha\beta$ as the area generated while α moves (remaining, of course, constantly parallel to itself) over a distance determined by β.*

Attaching this meaning to $p\varepsilon_1\varepsilon_2$ we think first of the point p moving over a distance determined by ε_1 generating a line, and then this line moving over a distance determined by ε_2 generating the square.

In the next two articles we will drop the factor p and think of the first factor as a line rather than as mere translation.

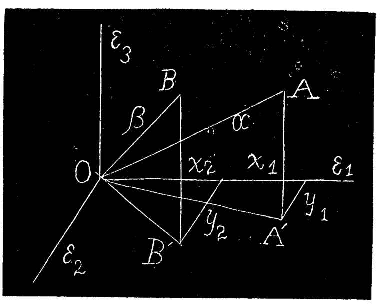
89. *Similarly let us seek the product of two vectors in space.* Let

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2 + z_1\varepsilon_3 \text{ and } \beta = x_2\varepsilon_1 + y_2\varepsilon_2 + z_2\varepsilon_3$$

where ε_1, ε_2, ε_3 are three unit vectors each at right angles to the other two. Then by 47,

$$[\alpha\beta] = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} [\varepsilon_1\ \varepsilon_2] + \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} [\varepsilon_2\ \varepsilon_3] + \begin{vmatrix} z_1 & x_1 \\ z_2 & x_2 \end{vmatrix} [\varepsilon_3\varepsilon_1].$$

Here it is evident that the first determinant coefficient is the area of the projection of the parallelogram whose adjacent sides are α and β on the plane $\varepsilon_1\varepsilon_2$, and that the other coefficients are the areas of the corresponding projections on the other planes. The above equation expresses then, that the area of the parallelogram whose sides are α and β is equal to the sum of its projections on the three coördinate planes (75).



90. *To find the product of three vectors, α, β, and γ.*

If ε_1, ε_2, ε_3 are three mutually perpendicular vectors, and

$$\alpha = x_1\varepsilon_1 + y_1\varepsilon_2 + z_1\varepsilon_3,\ \beta = x_2\varepsilon_1 + y_2\varepsilon_2 + z_2\varepsilon_3,\ \gamma = x_3\varepsilon_1 + y_3\varepsilon_2 + z_3\varepsilon_3.$$

$$[\alpha\beta\gamma]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}[\varepsilon_1\varepsilon_2\varepsilon_3] \qquad (45).$$

Hence if $[\varepsilon_1\varepsilon_2\varepsilon_3]$ is the volume of a cube each side of which is a linear unit, $[\alpha\beta\gamma]$ devotes the volume of the parallelopiped whose adjacent edges are α, β, γ, since, as is well known in analytic geometry, the determinant expresses the number of units of volume in the parallelopiped. In the *Ausdehnungslehre* attention must be paid to the *order* of the factors, *i. e.* to the order of generation. Thus (37) $[\alpha\beta]=-[\beta\alpha]$, and $[\alpha\beta\gamma]=-[\alpha\gamma\beta]$.

It is apparent from the preceding articles that the *Ausdehnungslehre* is especially well adapted for the investigation of propositions concerning the areas and volumes of rectilinear figures.

91. *To find the product of a posited point and vector.*

Let $p\rho$ and ε denote the point and vector. Following the areal interpretation given above this product should be a *line* whose length is ε and whose other dimension is the infinitesimal p, the whole fixed in position by the radius vector ρ.

Now $p(\rho+x\varepsilon)$ denotes any point on the line through $p\rho$. Then since

$$[p(\rho+x\varepsilon)\varepsilon]=[p\rho\varepsilon]+x[p\varepsilon\varepsilon]=[p\rho\varepsilon] \quad (34),$$

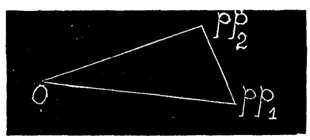
we see that the product of a posited point and a vector determine a line segment, *but this line segment may have any position on the vector through the given point.*

92. *To find the product of two posited points.*

Let $p\rho_1$ and $p\rho_2$ be two unit points. Then

$$[p\rho_1 . p\rho_2]=[p\rho_1(p\rho_2-p\rho_1)], \text{ since } [p\rho_1 . p\rho_1]=0. \quad (34)$$

But $[p\rho_1(p\rho_2-p\rho_1)]$ is the product of a point and a vector (78) which, by 91, is the line from the extremity of ρ_1 to that of ρ_2. Thus, *The product of two posited points is the line joining the first to the second.*



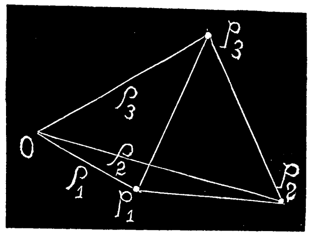
93. We have illustrated in the last article a principle which will be found to hold generally in the *Ausdehnungslehre*, viz., *That the product of posited quantities which have no common figure is some multiple of the connecting figure.*

94. *To find the product of three posited points.*

Let us use p_1, p_2, p_3 to denote the three unit points instead of $p\rho_1$, $p\rho_2$, $p\rho_3$ as heretofore. It will be understood when p is used to denote the posited point $p\rho$, that it stands for the complex quantity described in 76.

$$[p_1p_2p_3]=[p_1p_2 . p_3] \text{ (Rem. 13) } =[p_1p_2(p_3-p_1)], \text{ since } [p_1p_2p_1]=0 \text{ by } 43.$$

But $[p_1 p_2]$ is the line from p_1 to p_2 (92), and by 88 the product of a line $[p_1 p_2]$ and a vector $p_3 - p_1$ (78) equals the parallelogram whose adjacent sides are $[p_1 p_2]$ and $[p_1 p_3]$.



Thus the product of three given posited points is twice the area of the triangle whose vertices are the three given points. (93)

Let $p_1 + xp_2 + yp_3$ denote any point in the plane of $[p_1 p_2 p_3]$.

Now $[p_1 p_2 p_3]$ = twice area of triangle whose vertices are p_1, p_2, p_3; but we have

$$[(p_1 + xp_2 + yp_3)(p_2 - p_1)(p_3 - p_1)] = [p_1 p_2 p_3] \quad (22,\ 43).$$

This shows that *the value of the product remains the same whatever be the position of the triangle $[p_1 p_2 p_3]$ in the plane of these points.*

95. *To find the product of four posited points.*

Let p_1, p_2, p_3, p_4, represent four unit points. Then

$$[p_1 p_2 p_3 p_4] = [p_1 (p_2 - p_1)(p_3 - p_1)(p_4 - p_1)] \quad (43)$$

$$= 6 \times \text{tetraedron whose vertices are } p_1,\ p_2,\ p_3,\ p_4, \text{ by } 90.$$

Let $p_1 + xp_2 + yp_3 + zp_4$ be any point whatever. Then

$$[(p_1 + xp_2 + yp_3 + zp_4)(p_2 - p_1)(p_3 - p_1)(p_4 - p_1)] = [p_1 p_2 p_3 p_4]. \quad (22,\ 43)$$

Hence the product of four points in solid space is the same no matter where located.

96. We have used the terms "line" and "line segment" to denote a quantity whose length and the line in which it must lie are given but not its position in that line. Similarly we will use the terms "plane" and "plane segment" (See 74) to denote the corresponding areal quantity described in 94. Grassmann's terms for them are respectively "*Linientheil*" and "*Flächentheil.*" For the quantity described in 95 he uses the term "*Körpertheil.*"

97. *To find the sum or difference of two lines or two planes.*

Let $[p_1 p_2]$, $[p_1 p_3]$ be the lines, $[p_1 p_2 p_3]$, $[p_1 p_2 p_4]$, the planes, and let $p_3 + p_4 = 2p_s$, and $p_3 - p_4 = \varepsilon$. Then

$$[p_1 p_2] \pm [p_1 p_3] = [p_1 (p_2 \pm p_3)] = 2[p_1 p_s], \text{ or } [p_1 \varepsilon], \text{ a line.} \quad (82,\ 91)$$

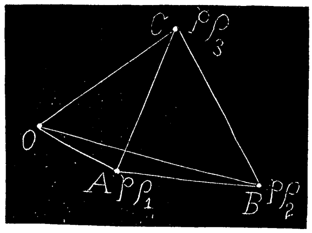
$$[p_1 p_2 p_3] \pm [p_1 p_2 p_4] = [p_1 p_2 (p_3 \pm p_4)] = 2[p_1 p_2 p_s], \text{ or } [p_1 p_2 \varepsilon], \text{ a plane.} \quad (94)$$

98. *To find the sum of the sides of a triangle.*

Let $p\rho_1$, $p\rho_2$, $p\rho_3$ represent the three vertices of a triangle. Then

$$[p\rho_1 \cdot p\rho_2] + [p\rho_2 \cdot p\rho_3] + [p\rho_3 \cdot p\rho_1] = [(p\rho_2 - p\rho_1)(p\rho_3 - p\rho_1)], \text{ since } [p\rho_1 p\rho_2] = 0.$$

Thus the sum of the three sides of a triangle equals the product of the vectors $p\rho_2 - p\rho_1$ and $p\rho_3 - p\rho_1$. This product differs from the expression for the area of the triangle (94) *by the absence of the first factor* $p\rho_1$. An interpretation of the expression given above for the sum of the sides which makes it equal to the area of the triangle may be had by thinking of $p\rho$ as a *generative* product of p and ρ. Using the period to denote generative multiplication, we have



$p.\rho_1.p\rho_2 = OAB$; $p.\rho_2.p\rho_3 = OBC$; $p.\rho_3.p\rho_1 = -OCA$.

$$\text{Thus, } OAB + OBC - OCA = p.\rho_1.p.\rho_2 + p.\rho_3.p\rho_2 + p.\rho_2.p\rho_3 = ABC.$$

Remark.—By Grassmann's formulas the sum of the sides of a triangle equals its area, for he treats a point as that which has position only, and considers that the product of two vectors alone equals the area described in 88. The use of the definition of a point given in 76 has the effect of making some of the theorems of this chapter depart in certain respects from Grassmann's. The writer thinks however that regarding a line as generated by a point in motion agrees well with Grassmann's conception of "generative" multiplication.

99. *If* $p_1, p_2, p_3, p_4, p_5, p_6$ *are six points,* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *are four vectors and* x *is any scalar, by the preceding articles we have the following conditions:*

(1) $p_1 = xp_2$ is the condition that points p_1 and p_2 coincide.

(2) $[p_1p_2] = x[p_3p_4]$, is the condition that the (unlimited) lines $[p_1p_2]$ and $[p_3p_4]$ coincide.

(3) $[p_1p_2p_3] = x[p_4p_5p_6]$ is the condition that the (unbounded) planes $[p_1p_2p_3]$ and $[p_4p_5p_6]$ coincide.

(4) $\varepsilon_1 = x\varepsilon_2$ is the condition that the vectors ε_1 and ε_2 are parallel.

(5) $[\varepsilon_1\varepsilon_2] = x[\varepsilon_3\varepsilon_4]$ is the condition that the planes of $[\varepsilon_1\varepsilon_2]$ and $[\varepsilon_3\varepsilon_4]$ are parallel.

100. A point is regarded as a space of the first order; an unlimited line as a space of the second order ; an unbounded plane as a space of the third order ; and solid space as a space of the fourth order. See 17 and 85. Since a vector may be regarded as a point at infinity, a vector also may be regarded as a quantity of the first order. See Chapter I.

101. Relative Products. A *Planimetric* product is a relative product whose factors are in a plane, or space of the third order. A Stereometric product is one whose factors are in a space of the fourth order (100).

102. *To find the planimetric product of two line segments* p_1p_2 and p_1p_3. We have

$$[p_1p_2.p_1p_3] = [p_1p_2p_3]p_1. \quad (67)$$

Here $[p_1p_2p_3]$ is a scalar (101, 61). Thus the product is the point of intersection multiplied by a scalar.

103. *To find the planimetric product of two parallel line segments* $[p_1p_2]$ *and* $[p_3p_4]$.

We have, by hypothesis, $p_3-p_4=x(p_1-p_2)$ (99, (4)). Then

$$[p_1p_2.p_3p_4]=[(p_1-p_2)p_2.(p_3-p_4)p_4]\ (49)=x[(p_1-p_2)p_2.(p_1-p_2)p_4]\ (\text{Hyp.})$$
$$=x[(p_1-p_2)p_2p_4](p_1-p_2)\ (67)=[(p_3-p_4)p_2p_4](p_1-p_2)\ (\text{Hyp.})$$
$$=[p_3p_2p_4](p_1-p_2)\ (49)=[p_2p_3p_4](p_2-p_1).\ (38)$$

Hence the product is the point at infinity (the vector p_2-p_1) which is the intersection of the two lines multiplied by the scalar $[p_2p_3p_4]$.

104. The last two articles illustrate a principle of general application in the *Ausdehnungslehre*, viz., *That the relative product of posited quantities which have a common figure is that common figure multiplied by a scalar.* See 93.

105. *To find the planimetric product of two lines and a posited point.*

Let $[p_1p_2]$ and $[p_1p_3]$ be the lines and p the point. Then

$$[p_1p_2.p_1p_3.p]=[(p_1p_2p_3)p_1.p]\ (13,\ \text{Rem.},\ 67)=[p_1p_2p_3][p_1p]$$

since $[p_1p_2p_3]$ is a scalar. (101, 61)

106. *To find the planimetric product of the three line segments* $[p_1p_2]$, $[p_1p_3]$, $[p_4p_5]$.

We have $[p_1p_2.p_1p_3.p_4p_5]=[p_1p_2p_3][p_1.p_4p_5]$ (67)$=[p_1p_2p_3][p_1p_4p_5]$ (55).

Corollary. *If the three line segments are the sides of a triangle we may write* $[p_2p_3]$ *instead of* $[p_4p_5]$. *Then the product is* $[p_1p_2p_3]^2$.

107. The following general principle is illustrated in the two preceding articles: *If at any time the product of factors combined in regular order from the left gives rise to a scalar or to a scalar times an extensive quantity, this scalar is to be regarded as a simple numerical factor, and the extensive quantity part of the product, if there is such, is to be combined with the remaining extensive factors, and so on.* Such products are described as *mixed, i. e.* as both progressive and regressive. (61)

108. *The stereometric products of a line and a point and of two lines are commutative; but that of a point and a plane is non-commutative.*

Let L_1 denote the line $[p_1p_2]$, L_2 the line $[p_3p_4]$, and P the plane $[p_2p_3p_4]$.

$$[L_1p_3]\equiv[p_1p_2.p_3]=[p_1p_2p_3]=[p_3.p_1p_2]\ (40,\ 55)\equiv[p_3L_1]\,;$$
$$[L_1L_2]\equiv[p_1p_2.p_3p_4]=[p_1p_2p_3p_4]\ (55)=[p_3p_4.p_1p_2]\ (40,\ 55)\equiv[L_1L_2]\,;$$
$$[Pp_4]\equiv[p_1p_2p_3p_4]=-[p_4.p_1p_2p_3]\ (40,\ 55)\equiv-[p_4P].$$

109. *To find the stereometric product of two non-incident plane segments* $[p_1p_2p_3]$ and $[p_1p_2p_4]$.

We have $[p_1p_2p_3 \cdot p_1p_2p_4]=[p_1p_2p_3p_4][p_1p_2]$ (67). See 104.

110. *To find the stereometric product of three plane segments* $[p_1p_2p_3]$, $[p_1p_2p_4]$, $[p_1p_3p_4]$ *which intersect in* p_1.

$$[p_1p_2p_3 \cdot p_1p_2p_4 \cdot p_1p_3p_4]=[p_1p_2p_3p_4][p_1p_2 \cdot p_1p_3p_4] \text{ (67)}=[p_1p_2p_3p_4]^2p_1 \text{ (67)}.$$

111. *To find the stereometric product of a plane segment* $[p_1p_2p_3]$ *and line segment* $[p_1p_4]$ *which do not lie in the same plane. Is the product commutative?*

We have $[p_1p_2p_3 \cdot p_1p_4]=[p_1p_2p_3p_4]p_1$ (67) (See 104).

Also, $[p_1p_4 \cdot p_1p_2p_3]=[p_1p_4p_2p_3]p_1$ (67)$=[p_1p_2p_3p_4]p_1$ (40).

112. *The stereometric product of two line segments* $[p_1p_2]$ *and* $[p_3p_4]$ *equals zero when and only when they lie in the same plane* (55, 95) ; *the stereometric product of two quantities of the first, second, or third orders, but not both at the same time of the second orders, equals zero when and only when the quantities are incident, i. e. when one falls in the space of the other ; as two coincident points, two plane segments if the planes coincide, a point and a line- or a plane-segment if the point lies in the line or plane, a line segment and a plane segment if the line lies in the plane.*

113. *Algebraic Curves and Surfaces. The equation of a variable point* p *which lies in the same straight line as* $[p_1p_2]$ *is* $[p\ p_1p_2]=0$. (112)

114. *The equation of a straight line* L *which passes through the intersection of* L_1 *and* L_2 *and lies in their plane is* $[L_1L_2L]=0$ *where* $[L_1L_2L]$ *is a planimetric product* (See 106).

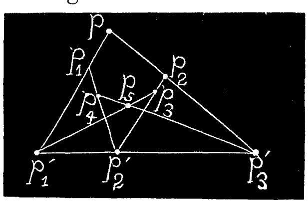
115. *If* $P_{n,p}$ *is a planimetric product of order zero which contains the variable point* p, n *times and besides only constant points and lines as factors, then* $P_{n,p}=0$ *is the point equation of an algebraic curve of the nth order, that is to say, the point* p *moves in an algebraic curve of the nth order.*

PROOF. Let p_1, p_2, p_3 be any three points in the plane. Then

$$p=x_1p_1+x_2p_2+x_3p_3$$

may be any point in the plane. Substituting this value of p in $P_{n,p}=0$ there results a homogeneous equation of the nth degree in x_1, x_2, x_3 whose terms are all of the form $Ax^a{}_1x^b{}_2x^c{}_3$ where $a+b+c=n$. A is the product of constant lines and points, and since this product is by hypothesis of the zeroth order, A is a constant. Regarding x_1, x_2, x_3 as trilinear coördinates, we see that $P_{n,p}=0$ now becomes an ordinary cartesian equation for a curve of the nth order.

116. As an example we give a proof of Pascal's Hexagram Theorem.



Let $p_1 \ldots p_5$ be five given points and p a variable point which moves so as to leave p'_2 on the line $[p'_1p'_3]$, p'_1, p'_2, p'_3 being defined by the following equations : $p'_1=[pp_1 \cdot p_3p_5]$; $p'_2=[p_2p_3 \cdot p_1p_4]$; $p'_3=[pp_2 \cdot p_4p_5]$. $[(pp_1 \cdot p_3p_5)(p_2p_3 \cdot p_1p_4)(pp_2 \cdot p_4p_5)]=0$ is the equation of a conic passing through the five given points. For it is of the second degree in p and is satisfied by putting p equal to any one of the five given points. By changing points into lines in the above we have Brianchon's Theorem.

[To be Continued.]

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS.

By GEORGE BRUCE HALSTED.

On the sixth of August at the Palais des Congrès in the Paris Exposition, was held the opening session of the second International Congress of Mathematicians. The president, Poincaré, is regarded as the greatest of living mathematicians. Among the vice presidents in attendance were Gordan, Lindeloef, Lindemann, Mittag-Leffler.

Representing Japan was Fujisawa ; Spain sent Zoel de Galdeano ; the United States, Miss Scott.

The president of the section of Arithmetic and Algebra was Hilbert ; of Geometry was Darboux, of Bibliography and History was Prince Roland Bonaparte. Among the most interesting personalities present may be mentioned Dickstein of Warsaw, Gutzmer of Jena, Hagen of Washington, Laisant of Paris, Langel of Golfe Juan, Lemoine of Paris, Delury of Toronto, Padoa of Rome, Shroeder of Carlsruhe, Sintsof of Yekaterinoslav, Stringham of Berkeley, Tannery of Paris, Vasiliev of Kazan. Whitehead of Cambridge.

Of the many important papers presented two may be selected for their general interest and the enthusiasm with which they were received.

These are : The Mathamatics of the Old Japanese School by Fujisawa, and The Problems of Mathematics by Hilbert.

Among other matters of extraordinary importance, Fujisawa showed his astonished audience that the Japanese had independently discovered the zero and by a mysterious coincidence used for it a circular symbol as did the Hindus and as do we. He showed that the Japanese had rectified the circle with an accuracy far exceeding Archimedes and only paralleled in our modern developments of pure mathematics. He showed that the Japanese had recognized $\sqrt{-1}$ the square root of minus one as a number, as a new unit, a neomon, and thus had reached the basis for the theory of the complex numbers.

This paper is epoch-making for the history of mathematics.

Hilbert's beautiful paper on the Problems of Mathematics shows that when a science progresses continuously we may from the problems which actually occupy it judge of its ulterior development. The existence of precise problems has a capital importance both for the progress of mathematics and for the work of each investigator.

Whence come the problems of mathematics ? It is experience that in each domain puts before us the primary problems (*e. g.* duplication of the cube, quadrature of the circle, etc.) In the later development of the science it is the mind which by logical reasonings (combination, generalization, specialization) creates itself problems new and fertile (*e. g.* problems of prime numbers). We say that a problem is solved when starting from a finite number of assumptions furnished by the problem itself we demonstrate the justness of the solution

by a finite number of deductions. This mathematical rigor which we require does not necessitate complicated demonstrations ; the most rigorous method is often the simplest and the easiest to comprehend.

The conceptions of arithmetic or those of analysis are not the only ones susceptible of rigorous treatment. Those of geometry and the physical sciences are equally so, provided that by means of a complete system of assumptions they are as well fixed as the conceptions of arithmetic.

When a problem presents serious difficultias, by what methods can we attack it ?

First by *generalization*, in attacking the problem considered to a group of questions of the same order. (*E. g.* Introduction of ideal numbers into the theory of algebraic numbers ; employment of complex paths in the theory of definite integrals).

Or else by *specialization*, in deepening the study of more simple analogous problems already solved.

The failure of attempts at the solution of a problem comes often from the problem being impossible to solve under the form given. Then we require a rigorous *demonstration of the impossibility*. (Parallel postulate, quadrature of the circle, algebraic solution of the equation of the fifth degree.)

We say that a conception exists from the mathematical point of view when the assumptions which define it are compatible, that is to say when a finite chain or system of logical deductions starting from these assumptions can never lead to a contradiction.

Mathematics in developing, far from losing its character of unique science, manifests it from day to day more clearly. Each real progress brings necessarily the discovery of methods more incisive and more simple, permitting to each geometer an access relatively facile to all the parts of our science.

The magnificent reception given by the President of France M. Loubet and his wife Madame Emilie Loubet in which the members of the Congress participated, was only surpassed in charm by the delightful entertainment given in our honor by Prince Roland Bonaparte.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

130. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Aow many balls 1 inch in diameter can be put in a cubical box 2 feet in the clear each way, putting in the maximum number ?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$31\times\frac{1}{2}\sqrt{2}+1=22.9201$.

Hence, we can put in 32 layers ; 16 layers of 576 in each layer and 16 layers of 529 in each. There still remains space enough for one more layer.

$23\times\frac{1}{2}\sqrt{3}+1=20.918$.

Hence, in this layer, we can put 3 rows of 24 balls each and 24 rows of 24 and 23 alternately, or 636 in the whole layer.

$$\begin{aligned} \therefore\ 16\times576&=\ 9216\\ 16\times529&=\ 8464\\ 1\times636&=\ \ 636\\ \hline \text{Total}&=18316 \end{aligned}$$

II. Solution by MARTIN H. SPINKS, Wilmington, Ohio.

Take the bottom layer and the rows in equilateral triangular form. The distance between the rows is .866 inch. The number of rows$=1+(23\div.866)=1+26=27$.

We then have 14 rows, 24 balls each, or 336 balls, and 13 rows, 23 balls each, or 299 balls each.

The bottom layer contains 336+299 or 635 balls. In the next layer we have 14 rows of 23 balls each or 322 balls, 13 rows of 24 balls each or 312 balls, in all 634 balls.

Distance between layers=.8162 inch.

Number of layers$=1+(23\div.8162)=28+1=29$.

Space left$=24-(1+28\times.8162)=.1464$ inch.

$\therefore$ We have 15 layers, 635 balls each, or 9525 balls
and 14 layers, 634 balls each, or 8876 balls

Total=18401 balls.

NOTE. Excellent solutions of problem 129 were received from H. C. Whitaker, P. S. Berg, G. B. M. Zerr, Martin Spinks, J. Scheffer, and O. S. Westcott. Mr. Gruber also furnished a non-rythmical solution. We think that his poetical solution is sufficiently clear and accurate as to be easily understood. The results of the various contributors differ slightly from Mr. Gruber's and from each other.

ALGEBRA.

106. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$$\frac{x^2+x}{y^2+y}=a\ ;\ \frac{x^2+y}{y^2+x}=b\ ;\ \text{find } x \text{ and } y.$$

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; L. B. FILLMAN, Chester, Pa.; and the PROPOSER.

By composition and division we have at once

$$\frac{x^2+x+y^2+y}{x^2-y^2+x-y}=\frac{a+1}{a-1}\ldots(1),\ \frac{x^2+y+y^2+x}{x^2-y^2+y-x}=\frac{b+1}{b-1}\ldots(2).$$

Multiplying (4) by (5),

$$xx'=yy'\left(y'\frac{a-b}{1+b}+1\right)\left(y\frac{a-b}{1+b}+1\right)=ayy'.\quad \text{From (3)}$$

$$\therefore yy'\left[\left(y'\frac{a-b}{1+b}+1\right)\left(y\frac{a-b}{1+b}+1\right)-a\right]=0.\ \ .(6).$$

The first two factors $y=0$, $y'=0$, give the roots $y=0$, $y=-1$; whence $x=0$, $x=-1$.

The remaining factor in (6) is clearly a quadratic for y, which yields the remaining two roots.

IV. **Solution by W. F. SHAW, 1600 Sabine Street, Austin, Tex.**

Clear of fractions and subtract

$$\begin{array}{l} x^2+x=ay^2+ay \\ x^2+y=by^2+bx \\ \hline x-y=(a-b)y^2+ay-bx \end{array}$$

Multiply first by b, second by a, and subtract.

$$\begin{array}{l} bx^2+bx=aby^2+aby \\ ax^2+ay=aby^2+abx \\ \hline (a-b)x^2+ay-bx=ab(x-y) \\ (a-b)y^2+ay-bx=(x-y),\ \text{above results.} \\ \hline (a-b)(x^2-y^2)=(ab-1)(x-y) \\ x+y=[(ab-1)/(a-b)] \end{array}$$

Combining this with one of the first equations the values of x and y are found to be

$$x=\frac{a+b+2ab\pm\sqrt{[4a(b^2+b+1)+(a+b)^2]}}{2(a-b)},$$

$$y=\frac{-(a+b+2)\mp\sqrt{[4a(b^2+b+1)+(a+b)^2]}}{2(a-b)}.$$

Combining the first two equations to eliminate one of the letters, a cubic results. Two roots having been found the third is quickly seen to be -1. The other set of values is then, $x=-1$, $y=-1$.

Solved in substantially the same manner by *J. M. BOORMAN*, Woodmere, N. Y.

GEOMETRY.

132. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

To draw a circle to cut two given circles orthogonally.

I. Solution by VIRGINIA CRAIG, Senior Classical Student, Drury College, Springfield, Mo.

Let a and b be the radii of circles with centers P and Q, $a+b+c$ be the distance between P and Q, and the line joining P and Q cut the circles at E and F. Divide the line c in the ratio $c+2b : c+2a$, and let the point of division be O. Construct a semi-circle on OP as diameter, said semi-circle intersecting circle whose center is P at L. With O as center, OL as radius, describe a circle intersecting circle whose center is Q at M. This described circle will cut the two given circles orthogonally.

Proof. OL is perpendicular to PL.

$\therefore$ Circles with centers O and P intersect orthogonally.

$OE+OF=c$ and $OE : OF=c+2b : c+2a$.

$$\therefore OE=\frac{c(c+2b)}{2(a+b+c)},\ OF=\frac{c(c+2a)}{2(a+b+c)}.$$

$$OL^2=OE(OE+2EP)=\frac{c(c+2b)}{2(a+b+c)}\left(\frac{c(c+2b)}{2(a+b+c)}+2a\right)$$

$$=\frac{c(c+2a)}{2(a+b+c)}\left(\frac{c(c+2a)}{2(a+b+c)}+2b\right)=OF(OF+2FQ).$$

Let OX be the tangent from O to circle with center Q. Then $OX^2=OF(OF+2FQ)$. But $OF(OF+2FQ)=OL^2=OM^2$.

$\therefore$ $OX=OM$, and as M is on circumference of circle with center Q, OX and OM coincide.

$\therefore$ OM is tangent to circle with center Q and perpendicular to MQ.

$\therefore$ Circles with centers O and Q intersect orthogonally.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; M. E. GRABER, Heidelberg University, Tifffn, O.; and J. SCHEFFER, A. M., Hagerstown, Md.

Any circle with center on the radical axis and radius equal to the tangent will evidently satisfy the conditions of the problem. The problem, then, resolves itself into the construction of the radical axis of two circles.

If the circles intersect, the radical axis is the common chord. If they do not intersect, draw any other circle cutting the given circles and draw their common chords. Their point of intersection is the radical center of the three circles. The radical axis required must pass through this point and must be perpendicu-

lar to the line joining the centers of the given circles. This offers no difficulty and the problem is solved.

Also solved by *ELMER SCHUYLER, HALLET E. McCLINTOCK;* and *CHARLES C. CROSS.* Professor Young furnished a neat diagram to accompany his solution of Problem 131.

CALCULUS.

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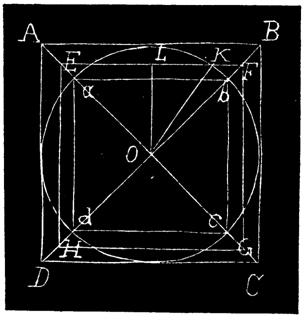
99. Proposed by **L. C. WALKER**, Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

The axis of three equal right circular cylinders intersect at right angles. Find the volume of the solid common to all.

Solution by **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School Chester, Pa.

Let $x^2+z^2=a^2$, $y^2+z^2=a^2$, $x^2+y^2=a^2$ be the equations of the cylinders. The projection of the intersection of the first and second on the plane xy is a square.



From $z=0$, to $z=\frac{1}{2}a\sqrt{2}$ the square decreases from $ABCD$ to $abcd$. The area of the projection common to the three cylinders between $z=0$ and $z=\frac{1}{2}a\sqrt{2}$ is eight times the area $LObKL$. From $z=\frac{1}{2}a\sqrt{2}$ to $z=a$ the common volume is the same as the volume common to the first and second.

Let $\angle LOK=\theta$. Area $EFGH=4xy=4(a^2-z^2)$.

Area $LOK=\frac{1}{2}a.OL\sin\theta=\frac{1}{2}ay\sin\theta$.

Area $KOb=\frac{1}{2}a^2(\frac{1}{4}\pi-\theta)$.

$\tan\theta=LK/OK=x/y=\sqrt{(a^2-y^2)}/y=z/\sqrt{(a^2-z^2)}$;

$\therefore \sin\theta=z/a$. $\therefore$ area $LOK=\frac{1}{2}z\sqrt{(a^2-z^2)}$, area $KOb=\frac{1}{2}a^2(\frac{1}{4}\pi-\sin^{-1}z/a)$.

$$\therefore V=8\int_0^{\frac{1}{2}a\sqrt{2}}[z\sqrt{(a^2-z^2)}+a^2(\tfrac{1}{4}\pi-\sin^{-1}z/a)]dz+8\int_{\frac{1}{2}a\sqrt{2}}^{a}(a^2-z^2)dz,$$

$$=8a^3(2-\sqrt{2}).$$

Also solved by *PROFS. ANDEREGG, SHERWOOD,* and *SCHMITT.*

100. Proposed by **B. F. FINKEL**, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

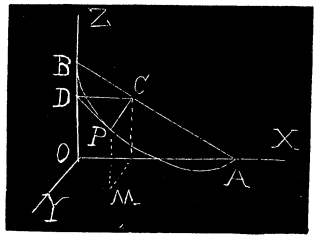
What is the volume bounded by the surface generated by the circumference of a circle whose diameter is the hypotenuse of a right-angled triangle whose base is b and altitude a, the plane of the circle being perpendicular to the plane of the triangle, the triangle and circle being rigidly connected, and the triangle revolving about its altitude a as an axis?

Solution by **F. ANDEREGG**, A.M., Professor of Mathematics, Oberlin College, Oberlin, O., **H. C. WHITAKER**, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; **G. B. M. ZERR**, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and the **PROPOSER**.

Let $AB=a$, $OA=b$, $AC=\sqrt{(a^2+b^2)}$; the coördinates of P and point in

the given circumference, (x, y, z); and the coördinates of M, the projection of P on the xy plane, (x, y).

Then $DC=bx/a$; $BC=\sqrt{(a^2+b^2)}\dfrac{x}{a}$;



$CA=\dfrac{\sqrt{(a^2+b^2)}(a-x)}{a}$; and $PC=\sqrt{(BC.CA)}$

$$=\frac{\sqrt{[(a^2+b^2)(a-x)x}}{a}.$$

$$\therefore\ DP^2=\frac{b^2x^2+(a^2+b^2)(a-x)x}{a^2}.$$

As the circle with radius DP moves parallel to itself and with its center on AB, it generates the volume required.

$$\therefore\ V=\frac{\pi}{a^2}\int_0^a[b^2x^2+(a^2+b^2)(a-x)x]dx=\frac{\pi a}{6}(a^2+3b^2).$$

MECHANICS.

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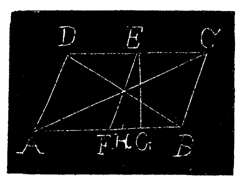
97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The side AB of the parallelogram $ABCD$ will be a principal axis at the point which divides the distance between the middle point and the foot of the perpendicular from the middle-point of the opposite side in the ratio 2 : 1. The principal moments of inertia about this point are $\frac{1}{3}mb^2\sin^2\beta$, $\frac{1}{36}m(3a^3+4b^2\cos^2\beta)$, where $\beta=\angle A$.

Solution by the PROPOSER.

Let $EH=c$, and let H be the origin, and lines through H parallel to EF, FB axes of coördinates.



$$\therefore\ \Sigma mxy=\rho\sin^2\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b y(x+y\cos\beta)dxdy$$

$=\frac{1}{6}mb\sin\beta(2b\cos\beta-3c)=0$ if HB is a principal. axis.

$\therefore\ c=\frac{2}{3}b\cos\beta$. But $FG=b\cos\beta$. $\therefore\ FH:HG=2:1$.

$$\Sigma my^2=\rho\sin^3\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b y2dxdy=\tfrac{1}{3}mb^2\sin^2\beta.$$

$$\Sigma mx^2=\rho\sin\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b(x+y\cos\beta)^2dxdy=\tfrac{1}{12}m(a^2+12c^2-12bc\cos\beta+4b^2\cos^2\beta)$$

$$=\tfrac{1}{36}m(3a^2+4b^2\cos^2\beta).$$

98. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, determine motion of spool and weight. [From problems in Mechanica at Harvard University.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let x and y be the horizontal and vertical parts of the string at any time t from the beginning of the motion, m the mass of the spool only, m' the mass attached to the thread, ϕ the angle of rotation of the spool in the time t, a and k the radius and radius of gyration of the spool, respectively, g the acceleration of gravity, T the tension in the thread, and F the friction.

The equation of motion of m' is $m'\dfrac{d^2y}{dt^2}=m'g-T$. . .(1).

For the linear and rotary motions of the spool,

$$m\frac{d^2x}{dt^2}=T-F. \; . \; .(2), \quad mk^2\frac{d^2\phi}{dt^2}=-Ta+Fa. \; . \; .(3).$$

Now $dx=ad\phi$. . .(4), and b being the initial length of free string, $x+y=b+a\phi$. . .(5).

From (4), $\dfrac{d^2\phi}{dt^2}=\dfrac{1}{a}\dfrac{d^2x}{dt^2}$. . .(6), and (3) is $mk^2\dfrac{d^2x}{dt^2}=-Ta^2+Fa^2$. . .(7).

From (2), $ma^2\dfrac{d^2x}{dt^2}=Ta^2-Fa^2$. . .(8).

(7) and (8) give $m(a^2+k^2)\dfrac{d^2x}{dt^2}=0$. . .(9), and $T=F$. . .(10).

From (5), $\dfrac{d^2x}{dt^2}+\dfrac{d^2y}{dt^2}=a\dfrac{d^2\phi}{dt^2}$. . .(11), which with (6) gives $\dfrac{d^2y}{dt}=0$. .(12).

(9) gives $\dfrac{d^2x}{dt^2}=0$. . .(13), and (6) then gives $\dfrac{d^2\phi}{dt^2}=0$. . .(14).

Hence, since the system starts from rest, the last three equations show that there is no motion afterwards.

AVERAGE AND PROBABILITY.

89. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

An inch auger-hole is bored at random through a six-inch sphere. Find the average volume of the auger-hole.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $x^2+y^2+z^2=R^2$ be the equation to the sphere.

Let $(x-a)^2+y^2=r^2$ be the equation to the auger.

Also let $\sqrt{[r^2-(x-a)^2]}=m$, $\sqrt{[R^2-r^2+a^2-2ax]}=n$.

From $a=0$, to $a=R-r$,

$$V=4\int_{a-r}^{a+r}\int_0^m\sqrt{[R^2-x^2-y^2]}dxdy=2\int_{a-r}^{a+r}mndx$$

$$+2\int_{a-r}^{a+r}(R^2-x^2)\sin^{-1}\left(\frac{m}{\sqrt{(R^2-x^2)}}\right)dx=2\int_{a-r}^{a+r}mndx$$

$$-\tfrac{2}{3}\int_{a-r}^{a+r}\frac{(3R^2x-x^3)(aR^2-R^2x+r^2x-a^2x+ax^2)dx}{mn(R^2-x^2)}$$

$$=2\int_{a-r}^{a+r}mndx+\tfrac{2}{3}\int_{a-r}^{a+r}\frac{nx^2dx}{m}-\tfrac{2}{3}\int_{a-r}^{a+r}\frac{x(R^2-x^2)dx}{mn}+\tfrac{4}{3}R^2\int_{a-r}^{a+r}\frac{mdx}{n}$$

$$-\tfrac{4}{3}R^2\int_{a-r}^{a+r}\frac{x(a-x)dx}{mn}-\tfrac{2}{3}R^3\int_{a-r}^{a+r}\frac{mdx}{(R+x)n}-\tfrac{2}{3}R^3\int_{a-r}^{a+r}\frac{mdx}{(R-x)n}.$$

Let $x=a-r\cos 2\theta$, $R^2-(a-r)^2=b^2$, $4ar/b^2=e^2$, $R+a-r=d$. $2r/d=c$, $R-a+r=h$, $-2r/h=c_1$, $\sqrt{[1-e^2\sin^2\theta]}=D$.

$$\therefore\ V=\tfrac{32}{3}br^2\int_0^{\frac{1}{2}\pi}D\sin^2\theta\cos^2\theta d\theta+\tfrac{16}{3}abr\int_0^{\frac{1}{2}\pi}D\sin^2\theta d\theta+\tfrac{4}{3}b(a-r)^2\int_0^{\frac{1}{2}\pi}Dd\theta$$

$$+\frac{32ar^3}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{D}+\frac{48ar^2(a-r)}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{D}+\frac{8ar(4R^2-3b^2)}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{D}$$

$$-\frac{4(a-r)(ab^2+2R^2r)}{3b}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{D}-\frac{16R^3r^2}{3bd}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)D}$$

$$-\frac{16R^3r^2}{3bh}\int_0^{\frac{1}{2}\pi}\frac{\sin 2\theta\cos^2\theta d\theta}{(1+c_1\sin^2\theta)D}.$$

$$\therefore\ V=\tfrac{4}{3}\Big[\frac{16br^2}{15e^4}(1-e^2+e^4)-\frac{4abr}{e^2}(1-2e^2)-\frac{8ar^3}{15be^6}(8+7e^2+8e^4)+b(a-r)^2$$

$$-\frac{24ar^2(a-r)}{3be^4}(1+e^2)-\frac{2ar(4R^2-3b^2)}{be^2}-\frac{4R^3r^2}{bcde^2}-\frac{4R^3r^2}{bc_1he^2}\Big]E(e,\ \tfrac{1}{2}\pi)$$

$$+\tfrac{4}{3}\Big[-\frac{8br^2}{15e^4}(2-3e^2+e^4)+\frac{4abr}{e^2}(1-e^2)+\frac{8ar^3}{15\,be^6}(8+3e^2+4e^4)$$

$$+\frac{12ar^2(a-r)}{3be^4}(2+e^2)+\frac{2ar\,(4R^2-3b^2)}{be^2}$$

$$-\frac{(a-r)\,(ab^2+2R^2r)}{b}-\frac{4R^3r^2}{bc^2de^2}(ce^2+e^2-c)-$$

$$\frac{4R^3r^2}{bc_1^2he^2}(c_1e^2+e^2-c_1)\Big]F(e,\ \tfrac{1}{2}\pi)+\frac{16R^3r^2}{3b}\Big[\frac{c+1}{dc^2}\Pi(e,\ c,\ \tfrac{1}{2}\pi)+\frac{c_1+1}{hc_1^2}\Pi(e,c_1,\tfrac{1}{2}\pi)\Big]$$

$\therefore\ V=AE(e,\ \tfrac{1}{2}\pi)+BF(e,\ \tfrac{1}{2}\pi)+C\Pi(e,\ c,\ \tfrac{1}{2}\pi)+C_1\Pi(e,\ c_1,\ \tfrac{1}{2}\pi)$, suppose.

Let $\dfrac{a^2+R^2-r^2}{2a}=x_1$; then from $a=R-r$ to $a=R+r$,

$$V_1=4\int_{x_1}^{R}\int_0^{\sqrt{(R^2-x^2)}}\sqrt{[R^2-x^2-y^2]}dxdy+4\int_{a-r}^{x_1}\int_0^{m}\sqrt{[R^2-x^2-y^2]}dxdy$$

$$=\pi\int_{x_1}^{R}\sqrt{[R^2-x^2]}dx+2\int_{a-r}^{x_1}mndx+2\int_{a-r}^{x_1}(R^2-x^2)\sin^{-1}\Big(\frac{m}{\sqrt{[R^2-x^2]}}\Big)dx.$$

$$\therefore V_1=\tfrac{1}{3}\pi[2R^3-(a-r)(2R^2+b^2)]+AE_0^{\theta_1}(e,\ \theta)+BF_0^{\theta_1}(e,\ \theta)+C\Pi_0^{\theta_1}(e,\ c,\ \theta)$$

$$+C_1\Pi_0^{\theta_1}(e,c_1,\theta),\text{ where }\theta_1=\tfrac{1}{2}\cos^{-1}\Big(\frac{a^2+r^2-R^2}{2a}\Big).$$

$$\therefore\ \triangle=\frac{\int_0^{R-r}Vda+\int_{R-r}^{R+r}V_1da}{\int_0^{R+r}da}.$$

A partial solution was given by *J. M. COLAW*.

MISCELLANEOUS.

83. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Segment's area of a circle whose radius is 5 inches is 28.56 square inches. Find the chord.

Solution by W.W. LANDIS, A.M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.; H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; COOPER D. SCHMITT, A.M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; G. B. M. ZERR, A.M., Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.

Let θ be the angle of the segment, c the chord.

$\therefore \frac{25}{2}(\theta-\sin\theta)=28.56$, $c=10\sin\frac{1}{2}\theta$.

But $\theta-\sin\theta=2.2848$.

$\therefore \theta=155^\circ\ 4'$ nearly. $\therefore c=9.764$ inches.

84. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathemalics. University of Tennessee, Knoxville, Tenn.

Prove that $e^\pi+e^{-\pi}=2[1+2^2][1+(\frac{2}{5})^2][1+(\frac{2}{5})^2]$. . .ad infinitum.

I. Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; J. O. MAHONEY, M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.; HARRY S. VANDIVER, Bala, Montgomery Co., Pa.; M. E. GRABER, Heidelberg University, Tiffin, O.; ELMER SCHUYLER, B. Sc., Professor of Mathematics and German, Boys' High School, Reading, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and PROPOSER.

From trigonometry we have the formula :

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\left(1-\frac{4\theta^2}{7^2\pi^2}\right).\ .\ .\ .\ .$$

In this replace θ by θi and we have

$$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\left(1+\frac{4\theta^2}{7^2\pi^2}\right).\ .\ .\ .\ .$$

Now let $\theta=\pi$, whence

$$\cosh\pi=(1+4)\left(1+\frac{4}{3^2}\right)\left(1+\frac{4}{5^2}\right)\left(1+\frac{4}{7^2}\right).\ .\ .\ .\ .$$

$$=[1+2^2][1+(\tfrac{2}{3})^2][1+(\tfrac{2}{5})^2][1+(\tfrac{2}{7})^2].$$

But $\cosh\pi=\frac{1}{2}[e^\pi+e^{-\pi}]$, whence the result.

II. Solution by MISS MARY M. BLAINE, A.M., Teacher of Mathematics, High School, Springfield, Mo., and MISS ALICE MADELEINE McKELDON, Graduate Student, University of Pennsylvania, Philadelphia, Pa.

To prove that $e^\pi+e^{-\pi}=2[1+2^2][1+(\frac{2}{3})^2][1+(\frac{2}{5})^2]$. . . .

We have (1) $\cos x=1-\dfrac{x^2}{2\,!}+\dfrac{x^4}{4\,!}-\dfrac{x^6}{6\,!}+$. . .

$$(2)\ \cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right).\ .\ .\ .\ .$$

[See Chrystal, Vol. II, page 329.]

Expanding this infinite product, and collecting,

$$\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\ldots\ldots=1-\frac{4x^2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\right]$$

$$+\frac{4^2x^4}{\pi^4}\left[\frac{1}{3^2}+\frac{1}{3^2\ 5^2}+-\right]-\frac{4^3x^6}{\pi^6}\left[\frac{1}{3^2\ 5^2}+-\right]+\ldots$$

$$=1-c_1x^2+c_2x^4-c_3x^6+\ldots(3).$$

Comparing (1), (2), and (3),

$$1-c_1x^2+c_2x^4-c_3x^6\ldots\ldots=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\ldots$$

$$\text{or}\left(1-\frac{4x^2}{\pi^3}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\ldots=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}\ldots(4).$$

Let $y^2=-x^2$. $\left(1+\frac{4y^2}{\pi^2}\right)\left(1+\frac{4y^2}{3^2\pi^2}\right)\ldots=1+\frac{y^2}{2\,!}+\frac{y^4}{4\,!}+\frac{y^6}{6\,!}+\ldots(5).$

Now, adding this exponential series, e^y and e^{-y},

$$\frac{e^y+e^{-y}}{2}=1+\frac{y^2}{2\,!}+\frac{y^4}{4\,!}+\ldots=\text{from (5)}\left(1+\frac{4y^2}{\pi}\right)\left(1+\frac{4y^2}{3^3\pi^2}\right)\ldots(6).$$

Substituting π for y in (6), we have,

$$e^\pi+e^{-\pi}=2[1+2^2][1+(\tfrac{2}{3})^2][1+(\tfrac{2}{5})^2]\ldots$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

132. Proposed by **WILLIAM SYMMONDS, A.M.**, Professor of Mathematics, Santa Rosa College, Sebastopol. Cal.

A road 60 feet wide crosses a square acre of land. The west line of the road passes through the southwest corner of the land, while the east line of the former passes through the northeast corner of the latter. What fraction of the land is included in the road?

133. Proposed by **COOPER D. SCHMITT, A. M.**, Professor of Mathematics, University of Tennessee. Knoxville, Tenn.

In Wentworth's Arithmetic he gives a formula $\frac{2}{4}\frac{1}{0}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

ALGEBRA.

121. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(x^5+y^5+z^5)^3+(x+y)^2=31$,

$(x^5+y^5+z^5)^3+(x+y+z)^3=729$,

$(x+y)^2+(x+y+z)^3=31$.

122. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A man buys a five per cent. ten-year bond at such a price as enables him to spend annually three per cent. upon his investment and by continually investing the residue of the annual interest and its increase annually at four per cent., at the end of term upon payment of his bond has his original investment. What price per $100 does he pay for the bond?

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

GEOMETRY.

148. Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics in Syracuse University, Syracuse, N. Y.

The condition that two triangles, abc, xyz, are similar is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix}=0,$$

and the condition that the triangle abc is equilateral is

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix}=0.$$

(Used in solving 130.)

149. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Given a conic and two circumscribing triangles of the conic; prove that the six vertices of the triangles are con-conic.

150. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Find the equation to a sphere cutting orthogonally four given spheres.

⁂ Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

CALCULUS.

112. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A sphere of radius r is pierced by a cylinder radius $\frac{1}{2}r$ so that the cylinder just grazes the center of the sphere. Find volume removed; the lateral surface and the spherical surface removed.

113. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

At what rate per unit of time are the roots of the equation $(x+px+q=0)$ changing, if $p=mq$ and q varies uniformly at the rate of 1/12 per unit of time, when $p=12$ and m remains constant?

⁂ Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

MISCELLANEOUS.

93. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Prove that $-(\sqrt{-1})^{\sqrt{-1}}=e^{(\sqrt{-1}-\frac{1}{2})\pi}$.

94. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The wall of a house, if its plane were extended, would cut the horizon at an angle$=\beta^\circ$ south of the true east point. The latitude of the place being$=\phi$, and the declination of the sun$=\delta$. When will the sun cease to shine through a window in that wall?

95. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, O.

"I enjoy here," said Goethe, "both good days and good nights. Often before dawn I am already awake, and lie down by the open window, to enjoy the splendor of the three planets, which are at present to be seen together, and to refresh myself with the increasing brilliancy of the morning-red." This was written in the summer of 1828 near Weimar. See Goethe's "Conversations with Eckermann," Bohn's Library, 1898, page 323.

What three planets are referred to?

*** Solutions to these problems should be sent to J. M. Colaw not later than Nov. 10.

EDITORIALS.

Prof. Leslie L. Locke has been elected Instructor in Mathematics in the Michigan Agricultural College, Ingram County, Michigan.

Our valued contributor, Dr. E. D. Roe, Jr., has been elected Professor of Mathematics in the Syracuse University, Syracuse, N. Y.

Our good friend, Prof. George B. M. Zerr, has been called to the chair of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Prof. Charles Scott Venable, LL. D., for the last five years Professor Emeritus of Mathematics at the University of Virginia, died on August 18 at his home in Charlottesville, Va. He was born in Prince Edward County, Va., on April 19, 1827. He received an academic education at the Hampden-Sidney College and the University of Virginia, being graduated at the former institution in 1842, and the latter in 1848. He continued his studies at Berlin in 1852, and at Bonn in 1854. Prior to 1861 he held professorships in the universities of South Carolina and Georgia, and in Hampden-Sidney College. In 1865 he was appointed professor of mathematics in the University of Virginia, and from 1870 to 1873 he was chairman of the Faculty. Professor Venable was the author of a series of text-books, including Arithmetic, Algebra, and Geometry.

BOOKS AND PERIODICALS.

The Theory and Practice of Interpolation, Including Mechanical Quadrature and other Important Problems Concerned with Tabular Values of Functions, with Requisite Tables. By Herbert L. Rice, M. A., Assistant in the office of the American Ephemeris and Professor of Astronomy in the Corcoran Scientific School, Washington, D. C. 4to cloth, 234 pages. Price, $3.50. Postage prepaid, $3.75. For sale by the author, United States Naval Observatory, Washington, D. C..

In the preparation of this treatise, the author has added to the technical literature of mathematics a work which gives a simple, practical, and comprehensive discussion of all that is useful concerning Differences, Interpolation, Tabular Differentiation, and Mechanical Quadrature, and has thereby rendered invaluable service to the practical computer. The work contains five chapters, an appendix, and eight tables. The first chapter treats of Differences; chapter II., of Interpolation; chapter III., of Derivatives of Tabular Functions; chapter IV., of Mechanical Quadrature; chapter V., of Miscellaneous Problems and Applications. The book is very beautifully printed with large type on heavy paper, and in every way makes a very good appearance. It is a work that not only every practical computer *must* have but every mathematician as well *should* have. B. F. F.

School Arithmetic, Primary Book. By J. M. Colaw, A. M., Associate Editor of the AMERICAN MATHEMATICAL MONTHLY, Monterey, Va., and J. K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pa., author of Table Book and Test Problems in Elementary Mathematics. 8vo. cloth, 271 pages. Price, 35 cents. Richmond, Va.: B. F. Johnson Co.

This new votary for public favor in the line of arithmetics has many points of excellence to recommend it. The authors in the preparation of this work have certainly proved that they understand the fundamental principles of sound pedagogy. The first thirty pages are devoted to giving the child an idea of comparing objects of different size. For example, on the first page we find this: "Cut splints the same length as A, B, and C," these letters referring to the pictures of sticks. "Show me the longest," "show me the next longest," "show me the shortest," "show me the next shortest," etc. By such exercises, which every child can master, it is led to simple work in numbers. The book is well written, beautifully and tastefully printed and illustrated. Teachers desiring a good primary arithmetic will do well to adopt this work. B. F. F.

An Elementary Physics for Secondary Schools. By Charles Burton Thwing, Ph. D., Professor of Physics in Knox College, formerly instructor in the University of Wisconsin, author of Exercises in Physical Measurement, Part I., Principles, Part II., Laboratory Exercises. 8vo. cloth, 371 pages. Price, $1.20. Boston; Benj. H. Sanborn & Co., Publishers.

During the past five years wonderful discoveries have been made in physics. These, though great as they seem, probably very feebly foreshadow those yet to be made during the first decade of the coming Twentieth century. These great discoveries, following each other in rapid succession, have made text-books written six or seven years ago obsolete. To put the principles of these discoveries clearly before the public, has necessitated the revision of all old works on physics and the writing of numerous new ones. Professor Thwing's work belongs to the latter class, and the object of its publication is to give the average student of a secondary school a book scientific, accurate, and up-to-date. The only unfavorable criticism that may be offered is that some of the illustrations lack artistic effect. B. F. F.

Higher Algebra. By John F. Downey, M. A., C. E., Professor of Mathematics in the University of Minnesota, 8vo., cloth and leather back, 416 pages. Introduction price, $1.50. New York and Chicago: American Book Co.

This work is written for the use of students in technical schools, colleges and universities. Some of the characteristic features of this new book are: (1) The logical demonstrations, each theorem or general principle being followed by a concise logical demonstration; (2) Short processes are used instead of the longer ones in common use; (3) The subject of Maxima and Minima is discussed in a fuller and more systematic way than usual; (4) Differentiation of algebraic and logarithmic functions are introduced. Numerous exercises and problems are given in all the different subjects. B. F. F

Holden's Elementary Astronomy. By Edward S. Holden, M. A., Sc. D., LL. D., former Director of the Lick Observatory. With over 200 illustrations. xiv+446 pages, 12mo. Price, $1.20. New York: Henry Holt & Co. 1899.

This new volume in the "American Science Series" is addressed especially to pupils who are studying the subject for the first time. The author has endeavored to overcome the difficulties of this study by a very full and gradual treatment of its elements. Elementary instruction in observation is an important feature. The book is one of exceptional interest and merit. J. M. C.

History of English Literature. By F. V. N. Painter, A. M., D. D., Professor of Modern Languages and Literature in Roanoke College. 697 pages. Boston: Sibley & Ducker. 1899.

This work is an eminently practical text-book. It is characterized by judicious selection and wise omission. Unusual prominence has been given to the writers of the nineteenth century. The literary map, the list of books of reference, and that of "books worth reading" add much to the interest and value of the work. It is well printed and beautifully illustrated. J. M. C.

Plane Trigonometry with Tables. By Elmer A. Lyman, Michigan State Normal School, and Edwin C. Goddard, University of Michigan. Price, $1.00. Boston: Allyn & Bacon. 1899.

The book includes those portions of Plane Trigonometry studied in high school and college classes. The general character of the demonstrations, the early introduction of inverse functions, the extended practice in the use of logarithms, the use of oral work to aid in fixing formulæ in the mind, and frequent reviews, are some of the distinctive features. The trigonometric equation has received careful treatment, and in the solution of triangles the division into cases has been abandoned. J. M. C.

Essentials of Plane and Solid Geometry. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 407 pages. Price, $1.25. Boston: D. C. Heath & Co. 1899.

In many of its features this work is similar to the author's Revised Plane and Solid Geometry, but important improvements have been introduced. The definitions and demonstrations are characterized by clearness, brevity, and accuracy. The book abounds in well-chosen and well-arranged exercises with excellent figures and suggestions. It ranks well with the very best books of its kind. J. M. C,

Mental Arithmetic. By Edward Weidenhamer, Ph. B. 173 pages. Cloth. Price, 35 cents. Harrisburg, Pa.: R. L. Myers &. Co. 1898, 1899.

This book begins with simple problems and proceeds by easy steps to those that are more difficult. The supply of problems is abundant, and in other respects this is a very satisfactory text. J. M. C.

The New Complete Arithmetic. By David M. Sensenig, M. S., and Robert F. Anderson, A. M., Instructors in Mathematics, State Normal School, West Chester, Pa. 427 pages. Price, 90 cents. Boston: Silver, Burdett & Co. 1900.

This book is designed to furnist to high schools, academies, and normal schools a complete treatise suitable for grades about to finish this branch of study. Under each subject examples of different types are solved in a manner calculated to direct attention to the logical steps involved. The very full treatment of business papers is a prominent feature. J. M. C.

The Elements of Arithmetic. By Ella M. Pierce, Supervisor of Schools, Providence, R. I. 149 pages. Price, 36 cents. Boston: Silver, Burdett & Co. 1900.

This book is intended for children of the third school year and covers the fundamental processes through numbers to one hundred. The lessons are simple and are well fitted to the age of the children for whom they are intended. J. M. C.

The Wooster Arithmetic—Grade I. By Lizzie E. Wooster. 112 pages. Cloth. Price, 25 cents. Topeka, Kansas: Crane & Co. 1899.

This book is intended for pupils in the first grade, and aims to do away with too much drill work upon the blackboard. The work advances by easy steps, and the number and variety of exercises are adequate. J. M. C.

The Elements of the Differential and Integral Calculus. By J. W. A. Young, Assistant Professor of Mathematical Pedagogy in the University of Chicago, and C. E. Linebarger, Instructor in Chemistry and Physics in the Lake View High School, Chicago. 410 pages. New York; D. Appleton & Co. 1900.

The present text is based closely upon the valuable German work of Professors Nernst and Schönflies which appeared in 1895. The fundamental principles and methods have been carefully treated in a manner that is in harmony with the more strict treatment possible in more extended treatises. The first chapter consists of an introduction to Analytic Geometry. Distinctive features are the exclusive use of the methods of limits, and the liberal use of illustrative examples from the natural sciences. The book has many points of excellence. J. M. C.

The Gospel According to Darwin. By Woods Hutchinson, A. M., M. D., of the University of Buffalo. 8vo. Paper cover. xii+241 pages. Price, 50 cents.

In this book the author has attempted to give merely a birds-eye-view of the influences affecting human hope and human happiness from the standpoint of that view of the universe and that attitude towards it which is best expressed by the term "Darwinism." In its pages are discussed in a very charming manner many of the themes that are most important to the human race. B. F. F.

We are indebted to Mr. C. M. Parker, Editor of *The School News and Practical Educator*, Taylorville, Ill., for copies of his valuable journal, containing a series of articles on Primary Number Work, by Prof. G. B. Longan.

J. M. C.

The following periodicals have been received since our last issue: *Journal de Mathématiques Élémentaires*, 15 Juillet 1900; *L'Intermédiaire des Mathématiciens*, Juillet 1900; *Notes and Queries*, September, 1900; *The Mathematical Gazette*, July, 1900; *The Educational Times*, August, 1900; *American Journal of*

Mathematics, July, 1900; *The Kansas University Quarterly*, *Proceedings of London Mathematical Society*, Vol. XXXI., *Periodico di Matematica*. J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine, edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., New York.

In the September *Review of Reviews* will be found a comprehensive treatment of the "Imperialism" issue, with particular reference to Mr. Bryan's Indianapolis speech. The editor's review of Mr. Bryan's Philippine propositions will be read with interest, alike by the adherents and the opponents of the Democratic candidate's policy. B. F. F.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

A journal for the school and the home. In it is given a resumé of every important event in the civilized world. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

Each number of this magazine is worth many times the yearly subscription price. In it culminates artistic excellence in magazine publications. B. F. F.

ERRATA.

Vol. VII, No. 4, page 104, second solution, the last line but two, for "37.7" read 31.5.

On line below the one just referred to, for "74.5" read 73.

On page 110, of the same number, for line 6, read $-\dfrac{8abr^2}{3c\sqrt{(a^2-r^2)}}F(e,\ \tfrac{1}{2}\pi)$.

On same page, instead of line 8, read $-\tfrac{8}{3}ab\sqrt{(a^2-r^2)}F(e,\ \tfrac{1}{2}\pi)$.

In No. 5, page 146, Problem 111, Calculus, for "of a hyperboloid or of a paraboloid," read, of a paraboloid or of a paraboloid.

THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. VII. OCTOBER, 1900. No. 10.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the August-September Number.]

CHAPTER VIII.

INNER PRODUCTS,—NORMAL SYSTEMS,—PROJECTION.

117. DEFINITION.—The Inner Product of two *units* of any order is the relative product of the first and complement of the second.

Thus the inner product of E and F is $[E|F]$.

Note.—Grassmann seems to have regarded the outer (52) and inner products as different in nature. But they both obey the laws of combinatory multiplication, the complement sign indicating a preliminary change to be made in the factor following it before it is combined with the other.

118. *The inner product of any two quantities is equal to the relative product of the first and complement of the second.*

PROOF.—Let $A=\alpha A_1+\ldots+\alpha_n A_n$, $B=\beta B_1+\ldots+\beta_n B_n$, where $A_1, \ldots,$ $B_1, \ldots,$ are units. Also for the moment let $\times$ signify the inner product.

$$\text{Then } [A \times B]\equiv[(\alpha_1 A_1+\ldots+\alpha_n B_n)\times(\beta_1 B_1+\ldots\beta_n B_n)]$$

$$=\Sigma\alpha_r\beta_s[A_r \times B_s]. \ (28)$$

Now since $A_1, \ldots, B_1, \ldots,$ are *units*, $[A_r \times B_s] = [A_r \mid B_s]$. (117)

Then $[A \times B] = \Sigma\alpha_r\beta_s[A_r \mid B_s] = \Sigma[\alpha_r A_r . \Sigma\beta_s \mid B_s]$ (28)

$$\equiv [A\Sigma\beta_s \mid B_s] = [A \mid \Sigma\beta_s B_s] \text{ (58) } \equiv [A \mid B].$$

119. *The inner product of two quantities of the same order is a number.* For, letting r denote the order of each factor, the complement of the second factor is of order n-r, and the product of the first factor which is of the order r and another which is of order n-r is of the nth order, *i. e.* is a pure number. (61)

Corollary.—*On account of the scalar value of the product, in this case* $[A \mid B] = [B \mid A]$.

120. *The inner product of two equal units is unity, while that of two different units of the same order is zero.*

Thus $[E_1 \mid E_1] = 1$ (57), $[E_r \mid E_s] = 0$. (43)

121. *If* $E_1, \ldots, E_n$ *are units of any order, but all of the same order, then*

$$[A \mid B] \equiv [(\alpha_1 E_1 + \ldots + \alpha_n E_n) \mid (\beta_1 E_1 + \ldots + \beta_n E_n)]$$

$$= \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_n\beta_n. \text{ (120)}$$

122. *If* $B = A$ *in* 121, *we get what is called* the inner square of A, which is denoted by A^2; thus we have

$$A^2 = \alpha_1^2 + \alpha_2^2 + \ldots \alpha_n^2.$$

123. Normal Systems.—Definition.—The numerical value of an extensive quantity A is defined as the positive square root of the inner square of A. This definition reminds one of the modulus in complex numbers.

124. Definition.—Two quantities (which do not equal zero) are said to be normal to each other if their inner product is zero. Two spaces are said to be every way (allseitig) normal to each other when each quantity in either space is normal to every quantity in the other space.

125. Definition.—A normal system of the nth order is a set of n numerically equal quantities of the first order of which each is normal to every other. If at the same time n is the order of the space, then such quantities constitute a *perfect* normal system. The numerical value of these n quantities is at the same time the numerical value of the system. Every normal system whose numerical value is unity is called a *simple* system.

126. Definition.—By Circular Alteration is meant that transformation of a system by which two quantities a and b of the system are transformed respectively into $xa + yb$ and $\pm(xb - ya)$, where $x^2 + y^2 = 1$. The circular alteration is said to be positive or negative according as $+$ or $-$ is taken in the double sign.

127. *By circular alteration any normal system is transformed into another normal system having the same numerical value.*

Proof.—Suppose a, b, . . . to be the quantities of a normal system. Then, by definition,

$$0=[a \mid b]=[a \mid c]=[b \mid c]=\ldots, \text{ and } a^2=b^2=c^2=\ldots$$

Let now a change into $a_1=xa+yb$ and b into $b_1=\pm(xb-ya)$ where $x^2+y^2=1$. We are to show that a_1, b_1, c . . . constitute a normal system. We have

$$a_1{}^2=(xa+yb)^2=x^2a^2+y^2b^2, \text{ since } [a \mid b]=0,$$

$$=(x^2+y^2)a^2=a^2, \text{ by hypothesis.}$$

Similarly, we can prove $b_1{}^2=b^2$.

Also, $[a_1 \mid b_1]=\pm[(xa+yb) \mid (xb-ya)]=\pm xy(b^2-a^2)=0$.

Finally, $[a_1 \mid c]=[(xa+yb)c]=x[a \mid c]+y[b \mid c]=0$.

Hence, by definition, a_1, b_1, c, . . . constitute a normal system.

128. *The combinatory product of quantities of a normal system is unaltered by positive circular alteration, and has its sign changed by negative circular alteration.*

Using the notation of 127, we have

$$[a_1b_1]=[(xa+yb)(xb-ya)]=x^2[ab]-y^2[ba] \ (34)=(x^2+y^2)[ab]=[ab].$$

129. *All the quantities of a normal system are independent.*

Proof.—Suppose a, b, c, . . . to be quantities of a normal system. Let us assume for the moment that they are not independent and that

$$a=\beta b+\gamma c+\ldots$$

We multiply both sides by $|a$. Then

$$a^2=\beta[b \mid a]+\gamma[c \mid a]+\ldots=0. \ (124)$$

But $a^2=0$ contradicts the hypothesis in 124. Hence the quantities of a normal system are independent.

130. *The system of the original units* (11) *is a perfect normal system whose numerical value is unity* (125).

Proof.—Let e_1, . . . e_n be the original units. Then (120)

$$e_1{}^2=e_2{}^2=\ldots=e_n{}^2=1, \text{ and } 0=[e_1 \mid e_2]=\ldots$$

131. Projection.—Definition.—If n is the order of the space considered, a_1, . . . a_n are independent quantities of the first order, A_1, A_2, . . . A_n are the

multiplicative combinations of these quantities of any one class, $A_1, \ldots A_m$ the multiplicative combinations of m of the same quantities $a_1, \ldots a_m$, and

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m + \ldots \alpha_n A_n$$

$$A' = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m.$$

A' is called the projection of A on the space $[a_1, a_2, \ldots a_m]$ by exclusion of the space $[a_{m+1}, \ldots a_n]$.

Remark.—We have introduced here for want of a better the geometrical term *projection* to translate *Zurückleitung*. Literally *Zurückleitung* means "leading back."

132. The projection A' of a quantity A on a space B by exclusion of the space C is

$$A' = \frac{[B.AC]}{[BC]}.$$

Proof.—Let the quantities be taken as in 131 and let

$$[a_1 \ldots a_m] = B, \; [a_{m+1} \ldots a_n] = C.$$

Then $[AC] = [(\alpha_1 A_1 + \ldots + \alpha_m A_m + \alpha_{m+1} A_{m+1} + \ldots \alpha_n A_n)C]$.

But since $A_1, \ldots A_m$ are the combinations formed out of $a_1, \ldots a_m$ and $A_{m+1}, \ldots A_n$ those out of $a_1, \ldots a_n$ which are not at the same time combinations out of $a_1, \ldots a_m$, then must each of the quantities $A_{m+1}, \ldots A_n$ contain at least one of the factors of $a_{m+1}, \ldots a_n$, and thus must have a factor in common with C. Therefore the terms

$$\alpha_{m+1} A_{m+1} C, \ldots \alpha_n A_n C$$

are each equal to zero. (43) Hence

$$[AC] = [(\alpha_1 A_1 + \ldots + \alpha_m A_m)C] = \alpha_1 [A_1 C] + \ldots + \alpha_m [A_m C].$$

$$\therefore \; [B.AC] = \alpha_1 [B.A_1 C] + \ldots + \alpha_m [B.A_m C].$$

Since now each of the quantities $A_1, \ldots A_m$ consists of factors which are contained in B, then is each of the same incident to B. Consequently, since the orders of B and C are together equal to n, by (72), we have

$$[B.A_1 C] = [BC] A_1, \ldots [B.A_n C] = [BC] A_n,$$

and therefore

$$[B.AC] = [BC](\alpha_1 A_1 + \ldots + \alpha_m A_m) = [BC] A'.$$

Now since $[BC]$ is a number, we get

$$A'=\frac{[B.AC]}{[BC]}.$$

133. *If the projections taken in the same sense of the terms of an equation replace those terms, the result is a true equation.*

PROOF.—Let Q be the space on which the projection is made, R that excluded and $[QR]=1$. Then if the given equation is

$$P=\alpha A+\beta B+\ldots$$

$$[PR]=\alpha[AR]+\beta[BR]+\ldots$$

$$\text{and } [Q.PR]=\alpha[Q.AR]+\beta[Q.BR]+\ldots$$

$$\text{or, } P'=\alpha A'+\beta B'+\ldots$$

where P', A', . . . are the *projections* of terms in the given equation.

134. DEFINITION.—The projection A' of a quantity A on a space B by exclusion of the space $|B$ is called the *normal* projection.

From 132 we have for the normal projection

$$A'=\frac{[B.(A\,|\,B)]}{B^2}.$$

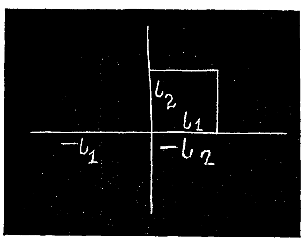
CHAPTER IX.

INNER PRODUCTS, NORMAL SYSTEMS, AND PROJECTION IN GEOMETRY.

135. Let ι_1 and ι_2 be two unit vectors constituting a simple normal system of the second order. Then by definition 125

$$\iota_1\iota_2=1, \text{ and } \iota_1\,|\,\iota_2=0,\ \iota_2\,|\,\iota_1=0.$$

We have also by definition of complement (57) $|\iota_1=\iota_2$ and $|\iota_2=-\iota_1$ since these values make $\iota_1\,|\,\iota_1=\iota_1\iota_2=1$, and $\iota_2\,|\,\iota_2=-\iota_2\iota_1=1$ (37). Also $||\iota_1=-\iota_1$ and $||\iota_2=-\iota_2$ (60).



Thus we see that taking the complement of a vector twice reverses it, *i. e.* revolves it through 180°, so that we are led to suppose that taking it once would revolve the vector through 90°. If this view of the complement can be shown to be consistent with the laws of the *Ausdehnungslehre*, we will adopt it.

We have introduced above the following equations whose geometrical interpretation we append to each.

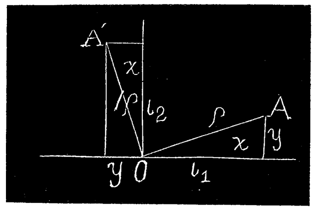
(1) $\iota_1\iota_2=1=$the unit of area (88).

(2) $|\iota_2=-\iota_1$, *i. e.* taking the complement of ι_2 revolves it in the positive direction, opposite to the motion of the hands of a watch, into $-\iota_1$.

(3) $\iota_1|\iota_2=\iota_1(-\iota_1)=0$ (34).

(4) Let $\rho=x\iota_1+y\iota_2$ be any vector in the plane. Then, 58,

$$|\rho=x|\iota_1+y|\iota_2=x\iota_2-y\iota_1.$$



The last value shows that $|\rho$ is OA', at right angles to OA. Thus here again taking the complement of a vector revolves it through 90° in the positive direction.

136. Comparing now the last part of the preceding article with 126-127 we see that the system whose units are ι_1 and ι_2 is transformed by circular alteration into that whose units are ρ and $|\rho$, provided $x^2+y^2=1$, which makes the tensors of the new vectors each equal to unity. Thus circular alteration turns each of the units through the same angle in the same direction.

137. *If ε_1 and ε_2 are any two vectors, $\varepsilon_1|\varepsilon_2=0$ is the condition that these two vectors are perpendicular to each other.*

For, $|\varepsilon_2$ denotes a vector perpendicular to ε_2 and $\varepsilon_1|\varepsilon_2=0$ denotes that ε_1 and $|\varepsilon_2$ coincide.

138. Let ι_1, ι_2, ι_3 be three unit vectors constituting a simple normal system of the third order. Then by Definition 125

$$\iota_1\iota_2\iota_3=1,\quad \iota_1|\iota_2=0,\quad \iota_1|\iota_3=0,\quad \iota_2|\iota_3=0.$$

We have also, by definition of complement (57),

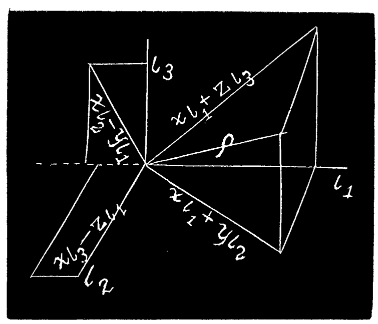
$$|\iota_1=\iota_2\iota_3,\ |\iota_2=\iota_3\iota_1,\ |\iota_3=\iota_1\iota_2\ ;\ ||\iota_1=\iota_1,\ ||\iota_2=\iota_2,\ ||\iota_3=\iota_3\ (60).$$

Thus we see (89) that the complement of a line is a plane, and the complement of a plane is a line.

Let $\rho=x\iota_1+y\iota_2+z\iota_3$=any line in space. Then, (58),

$$|\rho=x|\iota_1+y|\iota_2+z|\iota_3=x[\iota_2\iota_3]+y[\iota_3\iota_1]+z[\iota_1\iota_2]$$

$$=\frac{1}{x}[(x\iota_2-y\iota_1)(x\iota_3-z\iota_1)]\ (38,\ 34).$$



By 89 the right member equals the plane segment formed with $x\iota_2-y\iota_1$ and $x\iota_3-z\iota_1$.

Now ρ is perpendicular to each of these vectors and therefore perpendicular to their plane. For

$[(x\iota_1+y\iota_2+z\iota_3)|(x\iota_2-y\iota_1)]=0$, and

$$[(x\iota_1+y\iota_2+z\iota_3)\,|\,(x\iota_3-z\iota_1)]=0,$$

since, by hypothesis, $[\iota_1\,|\,\iota_2]=0$, etc.

Hence the complement of a vector is a plane perpendicular to it.

139. Projections.—*Let $\rho=x\varepsilon_1+y\varepsilon_2$ be given to find its projection on ε_1 and ε_2, respectively.*

Expressing ρ as the sum of its projections on ε_1 and ε_2, we have

$$\rho=\frac{[\varepsilon_1.\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}+\frac{[\varepsilon_2.\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\ (132)=\frac{[\rho\varepsilon_2]}{[\varepsilon_1\varepsilon_2]}\varepsilon_1+\frac{[\rho\varepsilon_1]}{[\varepsilon_2\varepsilon_1]}\varepsilon_2$$

since $[\rho\varepsilon_2]$, and $[\rho\varepsilon_1]$ are scalars in plane space.

140. *To express ρ as the sum of its projections on any three vectors ε_1, ε_2, ε_3.*

$$\rho=\frac{[\rho\varepsilon_2\varepsilon_3]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_1+\frac{[\rho\varepsilon_3\varepsilon_1]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_2+\frac{[\rho\varepsilon_1\varepsilon_2]}{[\varepsilon_1\varepsilon_2\varepsilon_3]}\varepsilon_3\ \text{(By 132. See 8).}$$

141. *To express p as the sum of its projections on any four points p_1, p_2, p_3, p_4.*

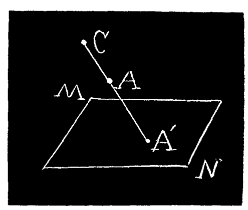
$$p=\frac{[p\;p_2p_3p_4]}{[p_1p_2p_3p_4]}p_1-\frac{[p\;p_3p_4p_1]}{[p_1p_2p_3p_4]}p_2+\frac{[p\;p_4p_1p_2]}{[p_1p_2p_3p_4]}p_3-\frac{[p\;p_1p_2p_3]}{[p_1p_2p_3p_4]}p_4\ (132).$$

See Articles 95 and 9. The substitution of $p\rho$ for p (76) in the last equation may serve to throw light on this case of point projection.

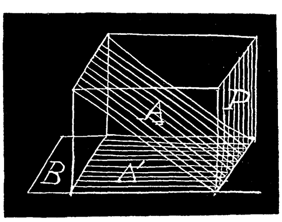
142. Since the formula of 132 is general in its application, the quantities in the equation of 140 may be all points, or all vectors, or all lines, or all plane vectors. In the equation of 141 the points may all be replaced by planes.

143. Following Hermann Grassmann Jr. in his notes to the *Ausdehnungslehre* of 1862, we will illustrate the formula of 132 by some geometrical examples. We suppose the quantities considered situated in solid space (4th order).

(1) *To find the projection A' of A on B by exclusion of C where A and C are points and B is the plane segment, MN.*



A' is the point where CA pierces B taken such that $A=nC+A'$. For, multiplying both members of $A=nC+A'$ by C, we have $[AC]=[A'C]$. Again multiplying B by both members of the last equation, we get $[B.AC]=[B.A'C]$. But $[B.A'C]=[BC]A'$ (72); whence $A'=[B.AC]\div[BC]$. By symmetry nC is the projection of A on C by exclusion of B.



(2) *To find the projection A' of the plane segment A by exclusion of the point C.*

We have $A=A'+P$ where P is in the plane passing through C and the intersection of A and B.

Proof follows lines of (1). Begin by multiplying through by C.

We have also P is the projection of A on C by exclusion of B as in (1).

144. Suppose $[q_2q_3q_4]$ denotes a plane and $[p_1p_2]$ a line. Then their stereometric product is a scaler times their point of intersection (111). Now let

$$[p_1p_2.q_2q_3q_4]=xp_1+yp_2=-[p_2q_2q_3q_4]p_1+[p_1q_2q_3q_4]p_2$$

by multiplying the members of the first equation by p_2 and p_1 in turn, thus getting values for x and y. Now multiply the members of the last equation by $|q_2$ and at the same time write $[q_2q_3q_4]$ as $|q_1$ (138). Then

$$[p_1p_2\,|q_1\,|q_2]=-[p_2\,|q_1][p_1\,|q_2]+[p_1\,|q_1][p_2\,|q_2]$$

$$\text{or } [p_1p_2\,|q_1q_2]=\begin{vmatrix} p_1\,|q_1 & p_1\,|q_2 \\ p_2\,|q_2 & p_2\,|q_2 \end{vmatrix} \text{ (55, 64).}$$

Putting $q_1=p_1$ and $q_2=p_2$ we have

$$[p_1p_2]^2=p_1^{\,2}p_2^{\,2}-[p_1\,|p_2]^2.$$

$p_1=1$ in the same equation, we have

$$[p_2\,|q_1q_2]=[p_2\,|q_2].|q_1-[p_2\,|q_1]\,|q_2.$$

This equation holds also when the p's are replaced by vectors.

145. Suppose $[p_1p_2p_3]$ denotes a plane and L a line. Then (111)

$$[p_1p_2p_3L]=xp_1+yp_2+zp_3=[p_2p_3L]p_1+[p_3p_1L]p_2+[p_1p_2L]p_3$$

by multiplying through by $[p_2p_3]$, $[p_3p_1]$, $[p_1p_2]$ in turn, thus getting values for x, y, z. Now for L put $|q_1q_2$ and multiply the members by $|q_3$. Then

$$[p_1p_2p_3.|q_1q_2.|q_3]=[p_2p_3\,|q_1q_2][p_1\,|q_3]$$

$$+[p_3p_1\,|q_1q_2][p_2\,|q_3]+[p_1p_2\,|q_1q_2][p_3\,|q_3],$$

$$\text{or, } [p_1p_2p_3\,|q_1q_2q_3]=\begin{vmatrix} p_1\,|q_1 & p_1\,|q_2 & p_1\,|q_3 \\ p_2\,|q_1 & p_2\,|q_2 & p_2\,|q_3 \\ p_3\,|q_1 & p_3\,|q_2 & p_3\,|q_3 \end{vmatrix} \text{ (55, 64, 144).}$$

In this equation planes may be substituted for points.

[To be Continued.]

TWENTY-FIFTH ANNIVERSARY OF PEDAGOGIC ACTIVITY OF VASILIEF.

By HUGH A. MILLER, Kazan, Russia.

Having read in your magazine [November, 1897] the biography of Alexander Vasilievitch Vasilief, and gathering from that article the estimation in which he is held, I thought that the American mathematical world might be interested to hear of the celebration by the Professor of his twenty-fifth year of pedagogic activity, at the ceremony in connection with which I had the honor to be present.

On the 12th of December, the day of the anniversary, *The Volga Messenger*, one of the principal papers of Kazan, wrote an article in praise of Vasilief. It spoke of him as a man of science, as a professor, and also as an active spirit in public life.

As a man of science Alexander Vasilievitch was possessed of remarkable knowledge in all branches of his subject, and moreover not content with possessing that knowledge, he had striven to impart it to his fellowmen by means of books, pamphlets, and public lectures ; of special value were his works on Lobatchefsky, Tchebeshef, and Auguste Comte.

As a professor he had taken the interests of all students to heart, and in his endeavor to improve their minds, he had inspired them with a love for science, and by his personal influence kept burning the sacred fire of enthusiasm.

As an active spirit in public life, Vasilief had always done his best to forward the interests of society at large, especially devoting his attention to the wants and requirements of the masses, and directing his efforts to the raising of their intellectual and economical condition.

His speeches on that subject had shown broad and progressive views, and had always met with the highest approval.

Telegrams from all parts of Russia and Europe had been pouring in since the eve, and by 12 noon on the day itself the inhabitants of Kazan began to arrive in order to congratulate Vasilief in person.

Among the first to come was a deputation of mathematical students, representing all four courses ; they bore an address bound in a handsome leather cover. The address was "To A. V. Vasilief on the occasion of the celebration of of his 25th year of pedagogic activity," and while expressing the inability of the students duly to appreciate his work as a man of science and as an active spirit in public life, hoped to convey to Vasilief their feelings towards him as a man and as a professor. It was indeed a stroke of good fortune which gave them the honor to congratulate the professor, and in doing so they felt in truth, that they were only the representatives of many former pupils, whose voice tho' unheard lent its approval to the sentiments expressed in the address. Students in all

corners of Russia owing their knowledge to Alexander Vasilievitch would remember him on this eventful day, and would join with them and the audience in wishing him a prolongation of his activity in the future.

The Rector of the University then paid his tribute of respect to the talents, energy, and work of Professor Vasilief. To him was due the high standard of mathematics maintained in the University, and also the revival of the memory of Lobatchefsky.

A deputation of the Physico-Mathematical Society then came forward to congratulate the learned professor; the spokesman was the dean of the mathematical faculty, and before beginning his address he handed Vasilief a medal commemorating his election as honorary member of the society of which Alexander Vasilievitch is already president.

The dean then expressed it as his opinion, that Vasilief's unparalleled activity in the Physico-Mathematical Society could hardly be sufficiently appreciated. Under his auspices the society which formerly existed as a branch of the Natural Science Society became an independent body, and it was due to his energy that this body had become famous in the world of science.

Its new existence had been marked by the appearance, thanks to Vasilief, of periodical editions of the Society's Magazine.

He had besides organized public lectures on the system of the "University Extension," and needless to say they were crowned with brilliant success. But his chief service to science consisted in his rescuing Lobachefsky from undeserved oblivion, and handing his name down to posterity. Highly appreciating all these services which left behind them ineffaceable traces the "Physico-Mathematical Society had resolved to express its gratitude by hanging the portrait of Alexander Vasilievitch in the Lobatchevsky Library, as the portrait of its chief founder and active supporter."

The numerous congratulatory letters and telegrams were then read out. They came from nearly all Russian mathematical professors, from all Russian mathematical societies, and also from numerous friends and admirers in various parts of Europe. Amongst the number was one from the St. Petersburg pedagogue Shochor-Troitsky: "We honor," wrote the professor, "not only the mathematician who thinks in a philosophical and historical manner, but a man of initiative who will never be forgotton in the history of Russian mathematics."

Kasan, Russia, January 13, 1900.

NOTE ON THE GENERAL EQUATION OF THE SECOND DEGREE.

By GEORGE R. DEAN, Missouri School of Mines, Rolla, Mo.

This paper is written for the benefit of the average sophomore who gets very little benefit out of the general discussion given in most text-books, and in most cases is bewildered with details.

Let the directrix be $x\cos\alpha + y\sin\alpha - p = 0$, focus (h, k), eccentricity e. Then the equation of the conic is

$$(1-e^2\cos^2\alpha)x^2 - 2e^2\sin\alpha\cos\alpha xy + (1-e^2\sin^2\alpha)y^2 - 2(h-e^2p\cos\alpha)x$$
$$+2(k-e^2p\sin\alpha)y + h^2 + k^2 - e^2p^2 = 0.$$

Comparing this with the equation

$$ax^2 + 2bxy + cy^2 + 2dx + 2ey + f = 0,$$

we have the six equations,

$$1-e^2\cos^2\alpha = an \ldots (1). \qquad (1-e^2\sin^2\alpha) = cn \ldots (2).$$
$$-2e^2\sin\alpha\cos\alpha = 2bn \ldots (3). \qquad h - e^2p\cos\alpha = -dn \ldots (4).$$
$$k - e^2p\sin\alpha = -en \ldots (5). \qquad h^2 + k^2 - e^2p^2 = fn \ldots (6).$$

Eliminating α between (1) and (2),

$$2-e^2 = (a+c)n, \text{ or } n = \frac{2-e^2}{a+c} \ldots (7).$$

Subtracting the square of (3) from the product of (1) and (2),

$$1-e^2 = (ac-b^2)n^2, \text{ or } n^2 = \frac{e^2-1}{b^2-ac} \ldots (8).$$

Since n^2 is positive, e^2-1 and b^2-ac must have the same sign.
Eliminating n between (7) and (8) and solving for e^2,

$$e^2 = \frac{[(a+c)^2+4(b-ac)] \pm (a+c)\sqrt{(a+c)^2+4(b^2-ac)}}{2(b^2-ac)}.$$

The value of e^2 being positive, the negative value is rejected.
Substituting in $n = \dfrac{2-e^2}{a+c}$, n is found.

From (1) and (2), $\cos 2\alpha = \dfrac{(c-a)n}{e^2}$.

From (3), $\sin 2\alpha = \dfrac{-2bn}{e^2}$.

These equations determine α without ambiguity. Substituting for α, e, n in the equations

$$h - e^2 p\cos\alpha = -dn,$$
$$k - e^2 p\sin\alpha = -en,$$
$$h^2 + k^2 - e^2p^2 = fn,$$

and solving for h, k, p, the curve is completely determined.

The solution of these last equations will be much simpler if the given equation is first transformed to the center, for we will then have

$$h = e^2p\cos\alpha,\quad k = e^2p\sin\alpha,\quad h^2+k^2-e^2p^2 = f'n,$$

f' being obtained by substituting the coördinates of the center in the left hand member of the given equation.

If a'=semi-major axis, b'=semi-minor axis, we have $a'=ep$, $b'=a'\sqrt{(1-e^2)}$.

Let us take the equation $3x^2+2xy+3y^2-16y+20=0$.

Transformed to centre $(-1, 3)$, $3x^2+2xy+3y^2-4=0$, we find

$$e^2 = \tfrac{1}{2},\quad n = \frac{2-\frac{1}{2}}{6} = \tfrac{1}{4},\quad \sin 2\alpha = -1,\quad \cos 2\alpha = 0.$$

$$\alpha = 135^\circ,\quad h = -\frac{p}{\sqrt{8}},\quad k = \frac{p}{\sqrt{8}},\quad p = \pm 2,\quad a' = \sqrt{2},\quad b' = 1.$$

In general the equations for h, k, p give two values of each quantity showing that the conic has two directrices and two foci.

If $e=1$, then $h - p\cos\alpha = -dn$, $k = -p\sin\alpha = -en$, $h^2+k^2-p^2 = fn$. The terms containing p^2 cancel, showing that the parabola has one directrix and one focus at infinity.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Continued from April Number.]

$$\therefore\ B_0 = \frac{1}{\pi}\int_0^{2\pi} \frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}}$$

$$=\frac{4}{\pi(1-e^2)^2}[2E(e,\tfrac{1}{2}\pi)-(1-e^2)F(e,\tfrac{1}{2}\pi)]\ldots\ldots(64).$$

$$B_1=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{3}{2}}}$$

$$=\frac{4}{\pi e(1-e^2)^2}[(1+4e^2)E(e,\tfrac{1}{2}\pi)-(1+2e^2)F(e,\tfrac{1}{2}\pi)]\ldots(65).$$

$$C_0=\frac{1}{\pi}\int_0^{2\pi}\frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{5}{2}}}$$

$$=\frac{4}{3\pi(1-e^2)^4}[8(1+e^2)E(e,\tfrac{1}{2}\pi)-(5-2e^2-3e^4)F(e,\tfrac{1}{2}\pi)]\ldots\ldots(66).$$

$$C_1=\frac{1}{\pi}\int_0^{2\pi}\frac{\cos\varphi d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{5}{2}}}$$

$$=\frac{4}{3\pi e(1-e^2)^4}[(1+14e^2-e^4)E(e,\tfrac{1}{2}\pi)-(1+6e^2-7e^4)F(e,\tfrac{1}{2}\pi)]\ldots(67).$$

$$D_0=\frac{1}{\pi}\int_0^{2\pi}\frac{d\varphi}{(1+e^2-2e\cos\varphi)^{\frac{7}{2}}}=\frac{4}{15\pi(1-e^2)^6}[2(23+82e^2$$

$$+23e^4)E(e,\tfrac{1}{2}\pi)-(31+51e^2-67e^4-15e^6)F(e,\tfrac{1}{2}\pi)]\ldots(68).$$

These expressions are also useful in finding the *disturbing* force between two planets.

Before dealing with general expressions we will find expressions containing the Elliptic Integral of the third order.

$$\int_0^{2\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\Pi(e,c,\tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots(69).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)-1]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\frac{1}{c}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}[F(e,\tfrac{1}{2}\pi)-\Pi(e,c,\tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(70).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(1-\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c}[(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)-F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots\ldots\ldots(71).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^2-2(1+c\sin^2\theta+1]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{c^2}\int_0^{\frac{1}{2}\pi}\frac{(c\sin^2\theta-1)d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[(c-e^2)F(e,\ \tfrac{1}{2}\pi)-cE(e,\ \tfrac{1}{2}\pi)+e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ldots\ldots\ (72).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(1-2\sin^2\theta+\sin^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[(c+1)^2e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-cE(e,\ \tfrac{1}{2}\pi)+(c-e^2-2ce^2)F(e,\ \tfrac{1}{2}\pi)]\ldots(73).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta-\sin^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^2e^2}[cE(e,\ \tfrac{1}{2}\pi)+(ce^2+e^2-c)F(e,\ \tfrac{1}{2}\pi)-(c+1)e^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(74).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^3}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^3-(1+3c\sin^2\theta+sc^2\sin^4\theta)]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[(3e^4-3ce^2+c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)-3e^4\Pi(e,\ c,\ \tfrac{1}{2}\pi)$$

$$-(2c^2+2c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots\ldots\ldots(75).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^4\theta-\sin^6\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[3e^4(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)+(2c^2-c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)$$

$$-(3e^4-3ce^2+2c^2-2c^2e^2+3ce^4)F(e,\ \tfrac{1}{2}\pi)]\ \ldots\ldots(76).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}+\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta\cos^2\theta-\sin^4\theta\cos^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3c^3e^4}[(3e^4+3c^2e^4-3ce^2+6ce^4-5c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)$$

$$-3e^4(c+1)^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(2c^2-4c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots(77).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\cos^4\theta-\sin^2\theta\cos^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{3\,c^3e^4}[3e^4(c+1)^3\Pi(e,\ c,\ \tfrac{1}{2}\pi)+(2c^2-7c^2e^2-3ce^2)E(e,\ \tfrac{1}{2}\pi)$$

$$-(3e^4+9c^2e^4-3ce^2+9ce^4-8c^2e^2+2c^2)F(e,\ \tfrac{1}{2}\pi)]\ldots\ \ldots\ldots..(78).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^8\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{[(1+c\sin^2\theta)^4-(1+4c\sin^2\theta+6c^2\sin^4\theta+4c^3\sin^6\theta)]d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{(1+c\sin^2\theta)^3d\theta}{\sqrt{(1-e^2\sin^2\theta)}}-\frac{1}{c^4}\int_0^{\frac{1}{2}\pi}\frac{(1+4c\sin^3\theta+6c^2\sin^4\theta+4c^3\sin^6\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[(8c^3+3c^3e^2+4c^3e^4-10c^2e^2-5c^2e^4+15ce^4-15e^6)F(e,\ \tfrac{1}{2}\pi)$$

$$+15e^6\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3+7c^3e^2+8c^3e^4-10c^2e^2-10c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)]..(79).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta\cos^2\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^6\theta-\sin^8\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[(8c^3-3c^3e^2-2c^3e^4-10c^2e^2+5c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)$$

$$-15e^6(c+1)\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-7c^3e^2-c^3e^4-10c^2e^2$$

$$+10c^2e^4+15ce^4-15ce^6-15e^6)F(e,\tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots\ldots\ldots(80).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^4\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^4\theta\cos^2\theta-\sin^6\theta\cos^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[15e^6(e+1)^2\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-13c^3e^2+3c^3e^4-10c^2e^2+20c^2e^4$$

$$+15ce^4)E(e,\ \tfrac{1}{2}\pi)+(8c^3-17c^3e^2+9c^3e^4-10c^2e^2+25c^2e^4$$

$$+15ce^4-30ce^6-15c^2e^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots..(81).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^6\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\sin^2\theta\cos^4\theta-\sin^4\theta\cos^4\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[(8c^3-23c^3e^2+23c^3e^4-10c^2e^2+35c^2e^4+15ce^4)E(e,\ \tfrac{1}{2}\pi)$$

$$-15e^6(c+1)^3\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-27c^3e^2+34c^3e^4-10c^2e^2$$

$$+40c^2e^4+15ce^4-15c^3e^6-45c^2e^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(82).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^8\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{(\cos^6\theta-\cos^6\theta\sin^2\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{15c^4e^6}[15e^6(c+1)^4\Pi(e,\ c,\ \tfrac{1}{2}\pi)-(8c^3-33c^3e^2+58c^3e^4-10c^2e^2+50c^2e^4$$

$$+15ce^4)E(e,\ \tfrac{1}{2}\pi)+(8c^3-37c^3e^2+74c^3e^4-10c^2e^2+55c^2e^4$$

$$+15ce^4-60c^3e^6-90c^2e^6-60ce^6-15e^6)F(e,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(83).$$

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^{10}\theta d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{c^5}\int_0^{\frac{1}{2}\pi}\frac{(1+c\sin^2\theta)^4d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{c^5}\int_0^{\frac{1}{2}\pi}\frac{(1+5c\sin^2\theta+10c^2\sin^4\theta+10c^3\sin^6\theta+5c^4\sin^8\theta)d\theta}{(1+c\sin^2\theta)\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{1}{105c^5e^8}[(48c^4+16c^4e^2+17c^4e^4+24c^4e^6-56c^3e^2-21c^3e^4-28c^3e^6$$

$$+70c^2e^4+35c^2e^6-105ce^6+105e^8)F(e,\ \tfrac{1}{2}\pi)-(48c^4+40c^4e^2$$

$$+40c^4e^4+48c^4e^6-56c^3e^2-49c^3e^4-56c^3e^6+70c^2e^4+70c^2e^6$$

$$-105ce^6)E(e,\ \tfrac{1}{2}\pi)-105e^8\Pi(e,\ c,\ \tfrac{1}{2}\pi)]\ldots\ldots\ldots\ldots(84).$$

As the foregoing are ample for illustration, we will proceed to other considerations.

[To be Continued.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

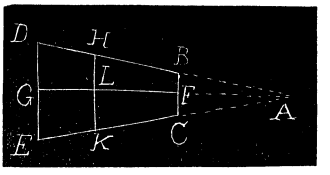
ARITHMETIC.

131. Proposed by M. A. GRUBER. A. M., War Department, Washington, D. C.

A right frustum of a cone whose radii of the bases are r and s, $r>s$, is to be divided into n parts of equal volume by sections parallel to the bases. What are the altitudes of the respective parts?

Solution by the PROPOSER.

Let $BCED$=section of given frustum through the centers of the bases.



Produce DB and EC until they meet in A; and draw AG perpendicular to DE. Then AF and AG= the respective altitudes of cones ABC and ADE; FG =altitude of given frustum; $DG=r$, and $BF=s$.

Draw HK parallel to DE so that the frustum with altitude FL is m/n part of the entire frustum.

Put $a=FG$; and let $x=AF$, $y_m=AL$, and $z_m=HL$.

The similar triangles ABF and ADG give $x:x+a=s:r$; or $x:a=s:r-s$.

$$\therefore\ x=\frac{as}{r-s},\text{ and } x+a=\frac{ar}{r-s}.$$

$$\therefore\ \frac{\pi ar^3}{3(r-s)}=\text{volume of cone } ADE;\ \frac{\pi as^3}{3(r-s)}=\text{volume of cone } ABC;$$

$$\tfrac{1}{3}\pi a(r^2+s^2+rs)=\text{volume of given frustum } BCED;$$

$$\frac{\pi am(r^2+s^2+rs)}{3n}=\text{volume of frustum } BCKH;$$

and $\dfrac{\pi a[mr^3+(n-m)s^3]}{3n(r-s)}$=volume of cone AHK=cone ABC+frustum $BCKH$.

From the similar volumes, cones ADE and AHK, we have

$$\frac{\pi ar^3}{3(r-s)}:\frac{\pi a[mr^3+(n-m)s^3]}{3n(r-s)}=\left(\frac{ar}{r-s}\right)^3:y_m{}^3.$$

$$\therefore\ y_m=\frac{a}{n(r-s)}\sqrt[3]{n^2[mr^3+(n-m)s^3]}.$$

$$\text{and } y_{m+1}=\frac{a}{n(r-s)}\sqrt[3]{n^2[(m+1)r^3+(n-m-1)s^3]}.$$

$$\therefore\ y_{m+1}-y_m=\frac{a}{n(r-s)}\{\sqrt[3]{n^2[(m+1)^3r^2+(n-m-1)s^3]}-\sqrt[3]{n^2[mr^3+(n-m)s^3]}\},$$

which is the general value for the respective altitudes of the n equal parts of the given frustum.

The limits of m are *zero* and n.

$$y_0=x=\frac{as}{r-s},\ \text{and}\ y_n=x+a=\frac{ar}{r-s}.$$

Hence as the altitudes of the equal parts diminish from s to r, $y_1-y_0=$ the greatest altitude and $y_n-y_{n-1}=$the least altitude.

The radius of the mth section is $z_m=\frac{1}{n}\sqrt[3]{n^2[mr^3+(n-m)s^3]}$.

Put $a=12$, $r=3$, $s=2$, and $n=4$.

Then $y_{m+1}-y_m=6\{\sqrt[3]{[2(19m+51)]}-\sqrt[3]{[2(19m+32)]}\}$.

Whence $y_1-y_0=6[\sqrt[3]{(102)}-4]=4.034$; $y_2-y_1=6[\sqrt[3]{(140)}-\sqrt[3]{(102)}]=3.121$; $y_3-y_2=6[\sqrt[3]{(178)}-\sqrt[3]{(140)}]=2.596$; and $y_4-y_3=6[6-\sqrt[3]{(178)}]=2.249$.

Also $z_1=\frac{1}{2}\sqrt[3]{(102)}=2.336$; $z_2=\frac{1}{2}\sqrt[3]{(140)}=2.596$; and $z_3=\frac{1}{2}\sqrt[3]{(178)}=2.812$.

Also solved in a very excellent manner by *G. B. M. ZERR*, and *J. SCHEFFER*.

ALGEBRA.

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107. Proposed by **CHARLES E. MYERS**, Canton, Ohio.

Given $xyz=18\ldots.(1)$; $x^2+y^2+z^2=33\ldots.(2)$; and $(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)(z^2-xy)=6561\ldots.(3)$; to find x, y, and z.

I. Solution by **M. A. GRUBER, A. M.**, War Department, Washington, D. C.

Expanding, uniting terms, and extracting square root of [3], we have

$$x^3+y^3+z^3-3xyz=81\ldots.[4].$$

Substituting $xyz=18$, and transposing, [4] becomes

$$x^3+y^3+z^3=135\ldots.[5].$$

Put $y=x+v$, and $z=x-v$.

Then, [1] becomes $x^3-xv^2=18\ldots.[6]$,

[2] becomes $3x^2+2v^2=33\ldots.[7]$,

and [5] becomes $3x^3+6xv^2=135\ \ldots[8]$.

From [6] and [8], we readily find $x=3$. Whence $v=\pm\sqrt{3}$.

$\therefore\ x=3$, $y=3\pm\sqrt{3}$, and $z=\mp\sqrt{3}$.

II. Solution by HARRY S. VANDIVER, Bala, Pa.

$xyz=a=18\ldots.(1)$.

$x^2+y^2+z^2=b=33\ldots.(2)$.

$(x^2-yz)^3+(y^2-xz)^3+(z^2-xy)^3-3(x^2-yz)(y^2-xz)(z^2-xy)=c=6561\ \ldots.(3)$.

Factoring (3) and using the customary notation for symmetric functions, we have

$$(\Sigma x)^2(\Sigma x^2-\Sigma xy)^2=\pm c$$

$$\text{whence } (\Sigma x)(\Sigma x^2-\Sigma xy)=\pm\sqrt{c}\ldots.(4).$$

Now assume that x, y and z are the roots of the cubic

$$v^3-mv^2+nv-p=0\ldots.(5),$$

and we are to determine m, n and p. From (1) we have $p=a$. Also we have identically,

$$(\Sigma x)^2=\Sigma x^2-2\Sigma xy\ldots.(5).$$

Then from (2), (4) and (5) we have the following equations to determine m and n:

$$m(b-n)=\pm\sqrt{c},\quad m^2=b-2n.$$

Eliminating and substituting, we find that the original system is equivalent to

$$\begin{cases} m^3+bm\mp2\sqrt{c}=0\ldots.(6) \\ v^3-mv^2+[b\mp(\sqrt{c}/m)]v-a=0\ldots.(7) \end{cases}$$

where $v=x$, y or z, and $\sqrt{c}$ has the same sign in both equations.

Of course, this system may be algebraically solved only in special cases, but substituting the given numerical values we obtain, writing (6) and (7) as two systems,

$$\begin{cases} m^3+33m+162=0. \\ v^3-mv^2+[33+(81/m)]v-18=0. \end{cases}$$

$$\begin{cases} m^3+33m-162=0. \\ v^3-mv^2+[33-(81/m)]v-18=0. \end{cases}$$

Both systems admit of being solved by Cardan's Method, giving 9 values for v in each—1 real and 8 imaginary.

Also solved by *H. W. KEATING*, Pittsburg, Pa., *ELMER SCHUYLER*, *J. SCHEFFER*, *G. B. M. ZEER*, and *COOPER D. SCHMITT*.

GEOMETRY.

133. **Proposed by P. C. CULLEN, Principal of Public Schools, Indianola, Neb.**

If the two bisectors, trisectors, quadrasectors, etc., of a triangle are mutually equal, show that the triangle is isosceles.

I. **Discussion by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.**

Eight different "proofs" of this theorem have thus far appeared in THE AMERICAN MATHEMATICAL MONTHLY. It is proposed in this note to point out some fallacious, or otherwise unsatisfactory reasoning in the demonstrations used.

I. Vol. II, No. 5, page 157.

The weak point in this is the apparent assumption that points O and N always fall within the side AB. Now, in the absence of definite knowledge to the contrary, we must assume that point O, for instance, may fall between A and B, on A, or in BA produced. So, either it must be proved that O can fall only between A and B, or else each of the other two cases must be considered, before the truth of the theorem is established.

II. Vol. II, No. 6, page 189.

Aside from the fact that this demonstration is unintelligible to the student merely of geometry, it is quite evident that there exist other relations between x and y than the one selected. Hence, barring the knowledge we may have from other sources, in this method there lurks the suspicion that the hypothesis of equal bisectors may yield also other than isosceles triangles. That is to say, "$x=y$" is not the only conclusion that can be drawn from the premises.

III. Vol. II, No. 6, page 190.

Here the fallacy is in assuming $BD.DC<BE.AE$, simply because $\angle A<\angle C$. With as much show of reason we might say, since $10<12$, and 5, a part of 10, $<9\frac{1}{2}$, a part of 12, therefore $5.5<9\frac{1}{2}.2\frac{1}{2}$.

IV. Vol. II, No. 6, page 160.

A most flagrant fallacy lies on the surface of the statement: "Now the right triangles AEL and ADK are similar, respectively, to AFN and AGM. But these last triangles are equal, and hence the triangles AEL and ADK are equal."

V. Vol. II, No. 6, page 160.

The fallacy here lies in the reasoning that the greater the arc, the greater the chord, which is necessarily true only when the arcs are less than a semi-circumference.

VI. Vol. V, No. 4, page 108.

This is the same "proof" as V above.

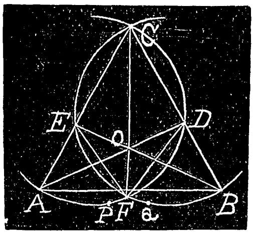
VII. Vol. V, No. 4, page 108.

The last statement in the first paragraph vitiates this demonstration. Because $\angle ACO>\angle ABO$, it does not follow that $AE>AF$; although it is true that if $\angle ACB>\angle ABC$, then $AB>AC$. But in the latter case, we have different criteria than in the former.

The only valid proofs among all the number in the MONTHLY are the following: VI, Vol. II, No. 6, page 189, and III, Vol. V, No. 4, page 109.

II. **Demonstration by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Let ABC be a triangle such that the bisectors AD and BE, of the angles A and B, respectively, are equal. About the triangles ADC and BEC describe circles. These circles are equal, since the chords AD and BE are equal, by hypothesis, and the arcs subtended by these chords, each being the measure of the angle C.



Therefore the chords BF and DF are equal, since each is subtended by the equal arcs BF and DF, which measure the angle FCB. In like manner, the chords AF and EF are equal. Therefore the triangles AFD and BFE are equal, and hence the perpendicular from F to AD is equal to the perpendicular from F to BD. Therefore OF bisects the angle AOB. Since O is the in-center, the line CO bisects the angle C, and, therefore, CO produced passes through the middle points of P and Q of the equal arcs AFD and BFE, respectively,—points equally distant from AD and BE, and, therefore, on opposite sides of the line OF. Hence the points P and Q must coincide with F, and the line AOF bisects both the angle ACB and AOC. Hence, since AO equals BO, and CO equals CO, and the angle ACO equals the angle BCO, the side AC equals the side BC, and the triangle is isosceles.

The same proof applies for the trisectors, quadrisectors, etc.

NOTE.—The above demonstration is adapted from a demonstration, in the *Educational Times*, London, England, by D. Biddle, editor of the Mathematical Department of that magazine. This seems to me to be a rigorous and direct demonstration and the simplest and most satisfactory that I have ever seen.

Mr. J. S. Mackey says, in the *Educational Times*, "A direct proof of this question will be found in the *London. Edinburgh, and Dublin Philosophical Magazine* (Fourth Series), Vol. XLVII, pages 354-7 (1874)." Mr. R. Tucker, in the same magazine, says, "This question was proposed as question 1907, in the *Lady's and Gentleman's Diary* for 1856, and is solved on page 58 (1857) by Messrs. T. T. Wilkinson, J. W. Elliott (the proposer), and analytically by others. Mr. Wilkinson returns to the problem in his 'Notæ Geometricæ' in the *Diary* for 1859, page 87. An historical note is added on page 88 which traces the question back to the *Nouvelles Annales* for 1842. Professor Sylvester drew attention to the subject in the *Philosophical Magazine* for November, 1852. Dr. Adamson further discusses the matter in the *Philosophical Magazine* for April, May, and June, 1853. The best article I know on question 1907 (*Diary*), appears in section 11 of Wilkinson's 'Notæ Geometricæ' in the *Diary* for 1860, pages 84-86, with a neat proof by Rev. W. Mason. I find that the above references are given in Dr. Mackay's *Euclid*, Page 108. In the key to this work, Dr. Mackay prints a proof by M. Descube."

Mr. W. J. Greenstreet, in the same journal, the *Educational Times*, says, "For this and the similar theorem for two symmedians, see *Intermediarie des Mathématiciens*, Vol. II (1895), pages 151, 325, If the external bisectors of A and B are equal, it does not follow that the triangles are isosceles. The data lead to $4Rr_1 = a^2 + bc$ in the triangle sides a, b, c (V. Mathesis, page 261, 1895)."

This theorem has been published in the MONTHLY three times already in the seven years of the MONTHLY's existence, and has been proposed for publication an innumerable number of times. We have given it considerable attention inasmuch as it has been considered by such noted mathematicians as Professor Sylvester. It is said that Dr. Todhunter tried to find a direct proof of it, but failed. Being the converse of the very simple theorem, *The bisectors of the base angles of an isosceles triangle are equal*, one hastily infers that the proof of it is quite easy. But such is not the case, as is readily seen when a simple and direct proof is attempted, and also from a study of the history of the theorem.

We hope that the above demonstration will be appreciated by our readers, and that in the future efforts, if any, will be made to give simpler proofs than this one.

Demonstrations were again furnished by G. B. M. Zerr, P. C. Cullen, George B. Birkhoff. We also received a few demonstrations that contained fallacies.

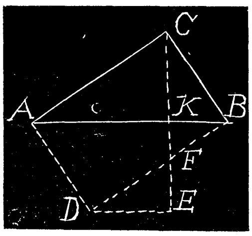
The fallacy in proof V, of Vol. II was also pointed out by Dr. E. S. Loomis of Cleveland, Ohio.

THE PYTHAGOREAN PROPOSITION.

Prof. D. A. Lehman, Baldwin University, Berea, Ohio, offers the following proof of the Pythagorean Proposition:

Given triangle ABC right angled at C, to prove $c^2=a^2+b^2$.

Draw AD parallel to CB, BD parallel to AC, CE perpendicular to AB, (cutting BD at F, and AB at K), HF parallel to CB, and DE parallel to AB.



Of the nine similar triangles thus made, let us consider three: (1) ABC, (2) FBC, (3) DEF.

In (1) and (2), $FB:a=a:b$; $\therefore$ $FB=a^2/b$. Also $CF:c=a:b$; $\therefore$ $CF=ac/b$.

Now $DF=DB-FB=b-a^2/b=\dfrac{b^2-a^2}{b}$.

In (3) and (1), $EF:a=\dfrac{b^2-a^2}{b}:c$; $EF=\dfrac{a}{c}\left(\dfrac{b^2-a^2}{b}\right)$.

$$ABC=\tfrac{1}{2}DBCA=\tfrac{1}{2}ab=\frac{c}{4}CE=\frac{c}{4}[CF+FE]=\frac{c}{4}\left[\frac{a\,c}{b}+\frac{a}{c}\left(\frac{b^2-a^2}{b}\right)\right]$$

$$=\frac{c}{4}\left[\frac{a\,c}{b}+\frac{a\,b}{c}-\frac{a^3}{bc}\right],\ i.\ e.,\ ab=\frac{ac^2}{2b}+\frac{a\,b}{2}-\frac{a^3}{2b},\ \text{or } c^2=a^2+b^2.$$

Dr. E. S. Loomis, Professor of Mathematics in the West Cleveland High School, contributes the following interesting results on Magic Squares:

Suppose in the above figure that the sides AC, BC, and AB are 4, 3, and 5, respectively. On these side construct squares containing 16, 9, and 25 unit squares. In the first row of unit squares along AC put the numbers

16, 6, 5, 19; in the next row above this, put the numbers 11, 13, 14, 8; in the next row above this, put the numbers 15, 9, 10, 12;
in the next 4, 18, 17, 7.

These numbers added horizontally, vertically, or diagonally give a sum of 46.

In like manner, place the following numbers as described above along CB beginning at C.

48, 47, 52
53, 49, 45
46, 57, 50

These numbers, when added horizontally, vertically, or diagonally, give a sum$=147$.

In the same way, place the following numbers in the unit square along AB beginning at A.

15, 16, 33, 30, 31
37, 22, 27, 26, 13
36, 29, 25, 21, 14
18, 24, 23, 28, 32
19, 34, 17, 20, 35

These, when added horizontally, vertically, or diagonally, give a sum$=125$.

Now $4\times46+3\times147=5\times125$, that is, the sum of all numbers in the hypotenuse—square=the sum of all numbers in the two leg-squares.

134. Proposed by J. C. GREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AB, CD passes through the center of the circle.

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio Univsrsity, Athens, Ohio.

Taking two of the lines as axes and the other two as $lx+my=1$......(1), $l'x+m'y=1$....(2), we find the two vertices of the quadrilateral on the coördinate axes $(0,\ 1/m)$, $(1/l',\ 0)$, and the middle of the diagonal of which these are the extremities $(1/2l',\ 1/2m)$....(3).

The coördinates of the intersection of (1) and (2) are

$$x'=\frac{m-m'}{l'm-lm'},\quad y'=\frac{l-l'}{l'm-lm'}\ldots.(4),$$

which and the origin are the extremities of the second diagonal. The middle of this diagonal is

$$\left(\frac{m-m'}{2(l'm-lm')},\quad \frac{l-l'}{2(l'm-lm')}\right)\ldots.(5).$$

Any conic touching the four lines is of the form

$$(ax+by-1)^2-2\lambda xy=0\ldots.(6).$$

For (6) to be tangent to (1) and (2),

$$\lambda=2(a-l)(b-m)\ldots.(7)\text{ or }\lambda=2(a-l')(b-m')\ldots.(8),$$

respectively. The center of (6) is given by

$$a(2ax-1)-\lambda y=0 \quad \ldots (9).$$

Eliminating a, b, and λ from (7), (8), and (9),

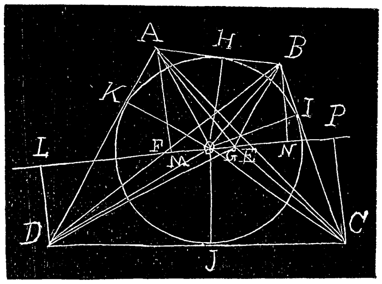
$$2ll'(m'-m)x+2mm'(l'-l)y+(l'm'-lm)=0 \ldots (10),$$

the locus of the centers of all conics inscribed in the given quadrilateral.

(10) is satisfied by (3) and (5).

II. Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ABCD$ be the quadrilateral; circumscribing the circle, center O; E, F the midpoints of AC, BD, respectively. Take G any other point in line LP through EF. Draw DL, AM, BN, CP perpendicular to LP. Then $\angle LDF=\angle NBF$, $DF=BF$.



$\therefore$ $DL=BN$. Also $\angle MAC=\angle PCA$, $AE=CE$.

$\therefore AM=CP$.

$\triangle BGC=\triangle BEC+\triangle BGE+\triangle CGE$.

$\triangle AGD=\triangle AED-\triangle AGE-\triangle GDE$.

But $\triangle BGE=\triangle GDE$, $\triangle CGE=\triangle AGE$.

$\therefore$ $\triangle BGC+\triangle AGD=\triangle BEC+\triangle AED$.

But $\triangle BEC=\frac{1}{2}\triangle ABC$, $\triangle AED=\frac{1}{2}\triangle ADC$.

$\therefore$ $\triangle BGC+\triangle AGD=\frac{1}{2}\triangle ABC+\frac{1}{2}\triangle ADC=\frac{1}{2}$ quadrilateral $ABCD$.

Similarly $\triangle AGB+\triangle DGC=\frac{1}{2}$ quadrilateral $ABCD$.

Therefore the straight line LP through EF is the locus of any point G which moves so that the sum of the triangles BGC and AGD is equal to the sum of the triangles AGB and CGD.

Now $\triangle AOB=\frac{1}{2}r.AB$, $\triangle BOC=\frac{1}{2}r.BC$.

$\triangle COD=\frac{1}{2}r.DC$, $\triangle DOA=\frac{1}{2}r.AD$.

Since $AB+DC=BC+AD$. $\triangle AOB+\triangle DOC=\triangle BOC+\triangle DOA=\frac{1}{2}$ quadrilateral $ABCD$.

$\therefore$ O must be on LP.

NOTE.—This theorem is exercise 66, on page 469, of Phill ps & Fisher's Elements of Geometry. We are quite sure that the ordinary Freshman would experience much trouble in effecting a demonstration. Professor Zerr has effected an excellent proof by Euclidean Geometry. The theorem was also proved by *COOPER D. SCHMITT* and *CHARLES C. CROSS*.

CALCULUS.

101. Proposed by WILLIAM FRED FLEMING, Denison, Tex.

A 24-inch joint of 6-inch stove pipe is compressed at one end to make it fit over an elliptical opening in a stove (for the escape of smoke). The ellipse has a major axis of 8 inches. What reduction is there in the solid contents of the stove pipe, assuming that its compressed shape may be generated by a 6-inch circle which passes uniformly from one end to the other and perpendicular to the axis of the pipe ?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

The perimeter of an ellipse semi-major axis a is

$$4a\int_0^{\frac{1}{2}\pi}\sqrt{1-e^2\sin^2\theta}\,d\theta=s.$$

In this problem $a=4$ and $s=6\pi$, the circumference of the pipe.

$$\therefore\ 6\pi=16\int_0^{\frac{1}{2}\pi}(1-\tfrac{1}{2}e^2\sin^2\theta-\tfrac{1}{8}e^4\sin^4\theta-\tfrac{1}{16}e^6\sin^6\theta-\text{etc.})d\theta$$

$$=8\pi(1-\tfrac{1}{4}e^2-\tfrac{3}{64}e^4-\tfrac{5}{256}e^6-\tfrac{175}{16384}e^8-\tfrac{441}{65536}e^{10}-\text{etc.})$$

Let $e=u$. Then $u+\frac{3}{16}u^2+\frac{5}{64}u^3+\frac{175}{4096}u^4+\frac{441}{16384}u^5+\ldots\ldots=1$.
By reversion of series $u=e^2=.8013$.
$\therefore\ b=a\sqrt{(1-e^2)}=4\sqrt{(.1987)}=1.783$ inches.

Let the pipe be compressed uniformly for the entire length. Also let πxy be the area of any elliptical section, and z the distance of this section from the circular end of the pipe.

Then $x=\dfrac{72+z}{24}$, $y=\dfrac{72-1.217z}{24}$.

$$V=\pi\int_0^{24}xydz=\frac{\pi}{576}\int_0^{24}(5184-15.624z-1.217z^2)dz.$$

$\therefore\ V=624.4568$ cubic inches.
V_1=volume before compression.
$\therefore\ V_1=9\pi\times24=678.5856$ cubic inches.
$\therefore\ V_1-V=54.1288$ cubic inches reduction in volume.

II. Solution by H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Denote the length of the pipe ($=24$ inches) by h, the semi-major axis ($=4$) of the ellipse by a, the semi-minor axis by b, the eccentricity by e, the radius of the circle ($=3$) by r. Then equating semi-perimeters :

$$\pi r = \pi a\left(1 - \frac{e^2}{2^2} - \frac{1^2.3.e^4}{2^2.4^2} - \frac{1^2.3^2.5.e^6}{2^2.4^2.6^2} - \right).$$

Solving by trial, $e=.895$, and $b=1.784$.

At a distance of z from the circular end of the pipe, the section is an ellipse, the semi-axes of which are found to be

$$\frac{hr-rz+az}{h} \text{ and } \frac{hr-rz+bz}{h}.$$

Hence the volume is

$$\int_0^h \pi\left(\frac{hr-rz+az}{h}\right)\left(\frac{hr-rz+bz}{h}\right)dz = \frac{\pi h}{6}(ar+br+2r^2+2ab) = 198.5\pi.$$

Hence the loss $=216\pi - 198.5\pi = 17.5\pi = 55$ cubic inches.

MISCELLANEOUS.

82. Proposed by A. H. BELL, Hillsboro, Ill.

Four spheres of equal radii $=r=5$, are in contact, and form a triangular pyramid. How large is the sphere that can be placed in the middle and be in contact with the four spheres.

Solution by J. W. YOUNG, Fellow in Mathematics, Cornell University, Ithaca, N. Y., and J. SCHEFFER, A. M., Hagerstown, Md.

Let A, B, C, B' (Fig. 1) be the centers of the four spheres. They evidently form the corners of a regular tetrahedron. Fig. 2 is a picture of a plane section of the pyramid of spheres, passed through the points $AB'L$, where L is the point of tangency of the two spheres (C, B).

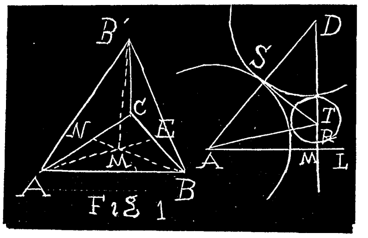


Fig. 1. . Fig. 2.

From Fig. 1,

$$\left.\begin{array}{l} AN/AM = \sin 60^\circ \\ AN/AB' = \cos 60^\circ \end{array}\right\}.$$

$\therefore AB'/AM = \tan 60^\circ = \sqrt{3} = \sec DAM$.

In Fig. 2, then, $\angle DAM$ is $\sec^{-1}\sqrt{3}$. It is clear that the required small sphere must have its center on DM and must touch both spheres (A, D). Let $\angle ADM = \theta$.

Then $\sin\theta = 1/\sqrt{3}$, $\cos\theta = \sqrt{\frac{2}{3}}$, $DT/r = \sqrt{\frac{3}{2}}$.

$\therefore DT = (r/2)\sqrt{6}$.

$\therefore RT = DT - r = (r/2)(\sqrt{6} - 2) =$ radius of small sphere.

$r=5$ gives $RT = 1.1238$.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

Solution by BENJ. F. YANNEY, A. M., Professor of Mathematics, Mount Union College, Alliance, O.

In THE AMERICAN MATHEMATICAL MONTHLY, No. 1, Vol. I, Dr. Dickson gives the following formulæ for finding lowest integers representing the sides of a right triangle: $m+\sqrt{(2mn)}$, $n+\sqrt{(2mn)}$, $m+n+\sqrt{(2mn)}$, in which m and n are shown to be integers, prime to each other, the one an odd square, the other twice any square, except such as would make m and n have a common factor.

We may represent, then, m and n by $(2r+1)^2$ and $2s^2$, respectively.

Now, if any one of the three sides of a rational right triangle is divisible by 5, so then also is their product; and, conversely.

The product of the three sides, represented by the above formulae, is $5m^2n+5mn^2+5mn\sqrt{(2mn)}+(m^2+n^2(\sqrt{(2mn)}$, the first three terms of which are evidently divisible by 5. Then, substituting in the last term for m and n their equals, $(2r+1)^2$ and $2s^2$, we have

$$2s(2r+1)[(2r+1)^4+4s^4]....(1).$$

Again, all possible integers may be represented by $5k+1$, $5k+2$, $5k+3$, $5k+4$, and $5k+5$.

It is plain that if $s=5k+5$, (1) will be divisible by 5, no matter what value r may have. So, too, if $r=5k+2$, no matter what s may equal. We still have to show that the last factor of (1) is divisible by 5 when s has any other value than $5k+5$, while at the same time r has any other value than $5k+2$.

Note that under these conditions all the literal terms of $(2r+1)^4$ and $4s^4$ are divisible by (5), while the numerical term of $(2r+1)^4$ always ends with 1, and that of $4s^4$ with 4; hence the numerical term of $(2r+1)^4+4s^4$ always ends with 5. Therefore, in any case, (1) is divisible by 5, which proves the proposition.

COROLLARY. By the same method, it may be proven that one of the numbers representing the legs of a rational right triangle must be divisible by 3. Dr. Dickson has shown that one must be divisible also by 4.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

134. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Science, Decorah Institute, Decorah, Iowa.

A certain piece of land is surrounded by a four-board fence, the boards being 16 feet long. The number of acres in the land equals the number of boards in the fence. How many acres in the land?

135. Proposed by NELSON L. RORAY, Brigdeton, N. J.

If 6 is one-half of 10, what part of 20 is 12? Also what part of 30 is 10?

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

ALGEBRA.

123. Proposed by **ELMER SCHUYLER**, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$$\left(\frac{1+x}{1-x}\right)^{\frac{1}{4}}+\sqrt{\frac{1-a}{1+a}}\sqrt[4]{\frac{1-x}{1+x}}=2\sqrt[4]{\frac{1-a^2}{(1+a)^2}},\text{ and }\sqrt{a^2-x^2}+x\sqrt{a^2-1}=a^2\sqrt{1-x^2}.$$

Haddon.

124. Proposed by **J. SCHEFFER**, A. M., Hagerstown, Md.

A certain quantity of alcohol diluted with water so that in one liter there are c liters of pure alcohol, is mixed n times successively with p times the quantity of alcohol diluted so that 1 liter contains a liter of pure alcohol. How much pure alcohol does one liter of the nth mixture contain?

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

GEOMETRY.

151. Proposed by **FRANK A. GRIFFIN**, Assistant in Mathematics, University of Colorado.

A point is at a distance of 1 inch, 2 inches, and $2\frac{1}{2}$ inches, respectively, from three corners of a square. Construct the square. Also solve for the general distances a, b, c.

152. Proposed by **ELMER SCHUYLER**, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Find a point in a given straight line, such that tangents drawn from it to two given circumferences shall make equal angles with the line. *Chauvenet.* (Four solutions.)

153. Proposed by **WILLIAM HOOVER**, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, Ohio.

If P, P', Q, Q' be the extremities of two chords of a conic section, and both chords pass through the point A, show that the sum of the squares of the reciprocals of AP, AP', AQ, AQ' is constant.

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

CALCULUS.

114. Proposed by **JOHN M. COLAW**, A. M., Monterey, Va.

If two concentric ellipses have equal axes inclined at an angle ω, their common area is

$$A=2ab\tan^{-1}\left(\frac{2ab}{(a^2-b^2)\sin\omega}\right).$$

115. Proposed by **F. P. MATZ**, M. Sc., Ph. D., Professor of Mathematics and Astronomy in Irving College, Mechahicsburg, Pa.

The axes of two right elliptic cylinders intersect at right angles, *major axes of the sections* are perpendicular. Supposing the axes to be $(A, B)>(a, b)$, what is the common volume?

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

MECHANICS.

103. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

If the sun were moved into the center of the earth's orbit, how much would the present length of the year be changed ?

104. Proposed by W. H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A tight roll of very thin and perfectly flexible oilcloth is placed upon a rough inclined plane, a portion of the cloth being unrolled and, extending from underneath the roll, is spread out smoothly upon the inclined plane below. The roll is then allowed to descend under the action of gravity, picking up the cloth as it goes. Determine the motion as far as possible.

*** Solutions of these problems should be sent to B. F. Finkel not later than Dec. 10.

DIOPHANTINE ANALYSIS.

85. Proposed by A. H. BELL, Hillsboro, Ill.

Given $x^2-85\frac{1}{4}y^2=5$. What is the value of x and y in whole numbers ?

86. Proposed by B. F. FINKEL, A. M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Prove that $x^2+1457\equiv0$ (mod. 2389) is insoluble.

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

MISCELLANEOUS.

96. Proposed by H. M. CASH, Love City, Guernsey County, Ohio.

A stick of timber is 12 feet long, 8 inches deep, and 3 inches wide at one end ; and 5 inches deep, and 12 inches wide at the other end. At what distance from either end should it be cut to divide it into two equal parts.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A spherical soap-bubble is electrified in such a way that the excess of the internal over the external air pressure is 2^π when the bubble is in equilibrium. How does the tension of the film vary with the electric density ?

*** Solutions of these problems should be sent to J. M. Colaw not later than Dec. 10.

EDITORIALS.

Prof. John B. Faught, Instructor in Mathematics in the Indiana University, has been elected Professor of Mathematics in the Northern State Normal School. Marquette, Michigan.

Prof. W. H. Carter, formerly Professor of Mathematics in Centenary College, Jackson, Louisiana, has been elected Professor of Mathematics in the University School for Boys, Stone Mountain, Georgia.

We are grieved to announce the death of H. T. J. Ludwig, Professor of Mathematics in the North Carolina College, Mt. Pleasant, North Carolina, which occured July 28th. Professor Ludwig was a regular contributor to the *Mathematical Visitor*, edited and published by Dr. Artemas Martin, to which he contributed some excellent solutions, and has been a subscriber to the MONTHLY from the beginning.

BOOKS AND PERIODICALS.

School Arithmetic.—Advanced Book. By John M. Colaw, A. M., Associate Editor of THE AMERICAN MATHEMATICAL MONTHLY, Monterey, Virginia, and J. K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pennsylvania, author of Table Book and Test Problems in Elementary Mathematics. 8vo. Cloth and Leather Back, 442 pages. Price, 60 cents. Richmond, Va.: B. F. Johnson Publishing Co.

This book, we are told in the preface, was prepared with a view to meet the needs of progressive teachers in the best public schools. The work is modern yet conservative. The Inductive Method is applied in great measure throughout the book; new topics are introduced by carefully prepared questions and suggestions; Oral and Written Work are given with every appropriate subject; few rules are given and these are usually in the form of definitions; and Bank Practice is treated from the stand-point of modern banking principles, the information on the subject having been furnished by prominent bank officials. The book is well gotten up and the authors are to be congratulated on the arrangement and plan of the work. B. F. F.

Annals of Mathematics. Published under the auspices of Harvard University. Second Series, Vol. II, No. 1. Published Quarterly. Price, $2.00 per year in advance.

This number contains the following articles: Wiener's Theory of Displacement, with an Application to the Proof of Four Theorems by Chasles, by Arthur Sullivan Gale; The Complex of Axes of a Central Quadratic Surface, by Edward V. Huntington; Galois' Theory of Algebraic Equations, Part II, Irrational Resolvents, by James Pierpoint. B. F. F.

Popular Astronomy. Edited by Wm. W. Payne and H. C. Wilson, Northfield, Minn. Issued Monthly. Price, $2.50 per year in advance.

The October number contains, among other articles, an interesting paper by Wm. W. Payne, on The Planet Eros. Also Tables of Observation of Eros. B. F. F.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

The issue for October 27 contains a brief resume of the principal events of the week. This magazine will keep the busy man posted on matters pertaining to Art, Science and Invention, and Religion. B. F. F.

THE AMERICAN MATHEMATICAL MONTHLY.

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VOL. VII. NOVEMBER, 1900. No. 11.

ROBERT TUCKER.

By GEORGE BRUCE HALSTED.

ROBERT TUCKER was born at Walworth in Surrey on April 26th, 1832. His forefathers were from the Isle of Wight. When at the Woodard School at New Shoreham the temporary head of the school was his cousin, Henry Jacobs, Michel Fellow of Queen's College, Oxford, and it was by his advice that Robert became a candidate for a sizarship at St. John's College, Cambridge (Sylvester's College). The Johnian freshman entered upon his University career in the Michaelmas term of 1851.

In the matter of college examinations the *vivâ voce* and the "Seven Devils" were still in existence.

The Seven Devils was an examination paper in algebra consisting of seven problem-puzzles of the most trying description, in which the data mostly translated themselves into simultaneous equations in an appalling number of unknowns. In the mathematical vivâ voce Tucker remembers being asked to expand $\tan x$ in a series. In this cheerful mental exercise he succeeded even beyond the first term. Three years later he was promoted to a Foundation Scholarship, and in the Tripos list of 1855 he was ranked as a Wrangler.

This year was a notable one for Johnian successes; no less than ten Johnians appeared in the first class, including the Senior, second, and third Wranglers. With the second Wrangler, Leonard Courtney, now the Right Honourable Member of Parliament for Bodmin, Tucker took a walking tour, planned to include several of the English battle-fields, Bosworth, Nasby, Worcester, etc.

The tour was made in a costume of Courtney's own designing, which attracted such attention that at Oakham their private room was invaded by the excited populace under the conviction that they were Paddy Noon and Paddy Gell, two noted prize-fighters who were expected.

In February, 1859, he took his M. A., and went as master to the school of J. A. Wall at Portarlington. Among his pupils here was W. M. T. Morgan, who afterward took a brilliant degree at Trinity College, Dublin.

In 1865 University College School had need of a successor to the late G. C. DeMorgan as Mathematical Master, and chiefly through the warm support of his candidature by Todhunter, Mr. Tucker was chosen. In the same year was founded the London Mathematical Society, and in October Mr. Tucker was elected a member. This was soon followed by his election to the Council, and, in November, 1867, by his appointment to the Honorary Secretaryship. This office he holds to the present day, having now been sole editor of seven hundred numbers of the Proceedings of the London Mathematical Society. In April, 1866, he married Elizabeth, the only daughter of William Byles, of Freshwater. They have three daughters.

The year 1871 saw a new undertaking in the shape of the Association for the Improvement of Geometrical Teaching. Mr. Tucker was Local Secretary for London, and subsequently Honorary Secretary and Vice President.

He has contributed to the mathematical columns of the *Educational Times* with scarcely an intermission from 1863. His memorial and biographical notices of Gauss, Chasles, Spottiswood, and Hirst together with reviews of many mathematical works, may be found in *Nature.* For Cayley's paper on Sylvester in that journal Mr. Tucker supplied the biographical details, with which it is important to note that Sylvester was satisfied, as he called upon Mr. Tucker and struck out only one insignificant detail.

As a geometer. Mr. Tucker is widely known as one of the creators of "Recent Geometry," or the modern Geometry of the Triangle, the Lemoine-Brocard Geometry. His name in this regard finds itself in honorable association with those of Brocard, Lemoine, and Neuberg.

There is a family of circles now universally known as "Tucker's Circles." If there be two triangles with parallel sides, their vertices upon copunctal straights, crossing on their common symmedian point. Then the six intersections of their sides are concyclic on a "Tucker's Circle."

Though the majority of University College School boys went in for engineering or a London University Degree yet the showing of Mr. Tucker's pupils at Cambridge is still very remarkable, witness the following:

1866. Ogle, 29th Wrangler.

1867. Puller, 19th Wrangler.

1874. W. W. Rouse Ball, 2d Wrangler and First Smith's Prizeman. (This is the well-known historian of mathematics).

1875. Saunder, 14th Wrangler.

1877. Kikuchi, 19th Wrangler. (The first Japanese Wrangler). [Since

Professor of Mathematics in the Imperial University, Tokio, and now its president. Member of the Japanese House of Lords. Author of a Geometry in English and Japanese].

1878. Sargant, 7th Wrangler.

1878. Leverson, 15th Wrangler.

1879. Karl Pearson, 3rd Wrangler. [The celebrated writer on the mathematics of Evolution and Darwinism].

1883. Romer, H. S., 20th Wrangler. [Brother to Ld. Romer, Senior Wrangler in 1863].

1885. Berry, A., Senior Wrangler and 2d Smith's Prizeman.

1886. Hooker, J. H., 29th Wrangler.

1887. Norris, J. R., 16th Wrangler.

1890. Bennett, G. T., Senior Wrangler and First Smith's Prizeman.

1890. Vaughan, A., Bracketed 3rd Wrangler.

1892. Kirby, S. F., 12th Wrangler.

1892. Clay, R. G., 21st Wrangler.

1895. Schroder, H. M., 29th Wrangler.

F. W. Frankland, the eminent writer on non-Euclidean Geometry (now an Actuary in New York) was a favorite pupil.

Lady Gwendolen Cecil, daughter of Lord Salisbury, read three seasons with Mr. Tucker, showing fine mathematical abilities.

A mere list of the writings of Mr. Tucker would give no adequate idea of their value. We may mention almost at random his Appendix to the Proceedings of the London Mathematical Society No. 279 containing Conjugate "Tucker" Circles. The Index to the Proceedings gives the titles of fifteen papers, ending with "Some Properties of Two Tucker-Circles."

In 1883 Mr. Tucker rediscovered Lemoine's circle and wrote a paper on it in the Quarterly Journal under the title: The "Triplicate-Ratio" Circle; which may be said to have started the general English interest in this new development of geometry.

Mr. Tucker did a fine piece of work in editing Clifford's Papers and his Dynamics Part II.

All the world will rejoice that as a result of a petition sent in to Mr. Balfour by the principal mathematicians of England, the Queen has granted Mr. Tucker a Civil List pension of forty pounds a year. The money is little, but such recognition of services must be highly gratifying to a loyal Briton.

DIAGRAMMATIC PROOF OF THE CONDITION OF FUNCTIONALITY IN COMPLEX FUNCTIONS.

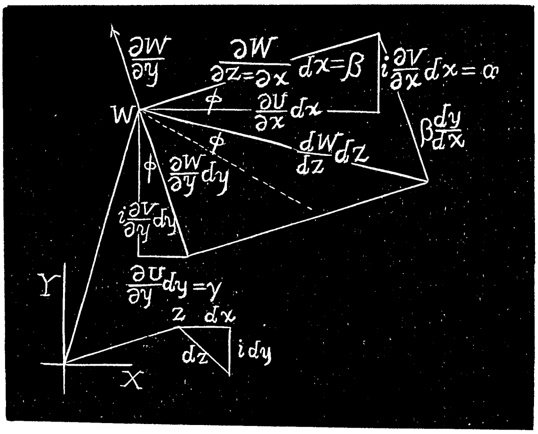
By A. LATHAM BAKER. C. E., Ph. D., University of Rochester, Rochester, N. Y.

1°. If the given complex function $W(z) = W(x+iy) = U + iV$ has a derivative we may write $\frac{dW}{dz} = w$, which phrased for the Argand Diagram is: *The triangle determined by dW, dz is similar to that determined by w, 1.*

At the point z, suppose z to take the increment $\partial z = dx$, causing in W, the change

$$\frac{\partial W}{\partial z = dx} dx = \frac{\partial U}{\partial x} dx + i \frac{dV}{\partial x} dx$$

lying at an angle, say ϕ, with $\partial z = dx$. This is shown in the diagram.



Suppose z had taken the increment dz, causing in W the change $\frac{dW}{dz} dz$, making with dz the same angle ϕ.

Since $\frac{\partial W}{\partial x} = \frac{dW}{dz} = \frac{w}{1}$, we have $\frac{dW}{\partial W} = \frac{dz}{\partial x}$ and $\frac{dW}{dz} dz$ will make with $\frac{\partial W}{\partial x} dx$ an angle whose secant is $\frac{dz}{dx} = \tan^{-1} \frac{dy}{dx}$.

Similarly when z takes the increment idy, $\frac{\partial W}{\partial z = dy} dy$ makes with the vertical the same angle ϕ. All these increments are shown in the diagram.

2°. Denoting the *lengths* of the lines in the diagram by Greek letters as shown, we get

$$\gamma = \beta \frac{dy}{dx} . \sin \beta = \beta \frac{dy}{dx} . \frac{\alpha}{\beta} = \alpha \frac{dy}{dx};$$

therefore

$$\frac{\gamma}{\alpha} = \left\{ \frac{\frac{\partial U}{\partial y} dy}{\frac{\partial V}{\partial x} dx} \right\} = \frac{dy}{dx},$$

or introducing the Argand elements,

$$\frac{\frac{\partial U}{\partial y}dy}{i\frac{\partial V}{\partial x}dx}=\frac{idy}{dx}.$$

In the same manner

$$\left|\frac{\partial V}{\partial y}dy\right|=\beta\frac{dy}{dx}\cos\phi=\beta\frac{dy}{dx}\frac{\frac{\partial U}{\partial x}dx}{\beta},=\frac{dy}{dx}\frac{\partial U}{\partial x}dx,$$

or in the Argand Diagram

$$\frac{i\frac{\partial V}{\partial y}dy}{\frac{\partial U}{\partial x}dx}=\frac{idy}{dx}.$$

These two results entail as the results of functionality (Cauchy's conditions).

$$\frac{\partial U}{\partial y}=-\frac{\partial V}{\partial x}. \quad \frac{\partial V}{\partial y}dy=\frac{\partial U}{\partial x}dx.$$

3°. Or we might have said, since $\frac{\gamma}{dy}=\frac{\alpha}{dx}$ or $\left|\frac{\partial U}{\partial y}\right|=\left|i\frac{\partial V}{\partial x}\right|$.

Whence multiplying $i\frac{\partial V}{\partial x}$ by i revolves it into coincidence with $\frac{\partial U}{\partial y}$, that is, (see paragraph 4°) $-\frac{\partial V}{\partial x}=\frac{\partial U}{\partial y}$.

Similarly, $\left|i\frac{\partial V}{\partial y}\right|=\left|\frac{\partial U}{\partial x}\right|$.

Whence, $i\frac{\partial V}{\partial y}=i\frac{\partial U}{\partial x}$, or $\frac{\partial V}{\partial y}=\frac{\partial U}{\partial x}$.

4°. Since $\frac{\beta}{\delta}=\frac{dx}{dy}$, in the same way $\left|\frac{\partial W}{\partial x}\right|=\left|\frac{\partial W}{\partial y}\right|$. But $\frac{\partial W}{\partial x}$ was produced by dividing $\frac{\partial W}{\partial y}dy$ by dy. In the diagram dy is essentially negative, and remembering that division is doing to the operand what was done to the divisor to produce unity, we revolve $\frac{\partial W}{\partial y}dy$ through 180°, and change in the proper ratio, giving $\frac{\partial W}{\partial y}$ as shown in the diagram. The same remarks apply to $\frac{\partial W}{\partial x}$, but here the angle is zero.

Evidently revolving $\frac{\partial W}{\partial x}$ through 90° gives coincidence with $\frac{\partial W}{\partial y}$, or

$$i\frac{\partial W}{\partial x}=\frac{\partial W}{\partial y},$$

which is Riemann's condition of functionality.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Continued from October Number.]

Let $e_1=\frac{e^2}{2\pm e^2}$, $e_2=\frac{e_1^2}{2\pm e_1^2}$, $e_3=\frac{e_2^2}{2\pm e_2^2}$, ... , $e_n=\frac{e^2{}_{n-1}}{2\pm e^2{}_{n-1}}$.

$$[(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R,$$

$$[(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_1,$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]^{\frac{1}{2}}=R_2,$$

$$[(1-e_3^2\sin^2 8\theta)(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]=R_3$$

. . . . . .

$$[(1-e_n^2\sin^2 2^n\theta)(1-e^2{}_{n-1}\sin^2 2^{n-1}\theta)\ldots.(1-e_3^2\sin^2 8\theta)$$

$$[(1-e_2^2\sin^2 4\theta)(1-e_1^2\sin^2 2\theta)(1\pm e^2\sin^2\theta)(1\pm e^2\cos^2\theta)]\ \ldots R_n.$$

But $R=[1\pm e^2(\sin^2\theta+\cos^2\theta)+e^4\sin^2\theta\cos^2\theta]^{\frac{1}{2}}=[1\pm e^2+\frac{1}{4}e^4\sin^2 2\theta]^{\frac{1}{2}}$

$$=\tfrac{1}{2}[(2\pm e^2)^2-e^4\cos^2 2\theta]^{\frac{1}{2}}=\tfrac{1}{2}(2\pm e^2)[1-e_1^2\sin^2(\tfrac{1}{2}\pi-2\theta)]^{\frac{1}{2}}.$$

Let $\frac{1}{2}\pi-2\theta=\varphi$. Then $d\theta=-\frac{1}{2}d\varphi$.

$$\therefore \int_0^{\frac{1}{2}\pi}Rd\theta=\tfrac{1}{4}(2\pm e^2)\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi}\surd(1-e_1^2\sin^2\varphi)d\varphi$$

$$=\tfrac{1}{2}(2\pm e^2)\int_0^{\frac{1}{2}\pi}\surd(1-e_1^2\sin^2\varphi)d\varphi=\tfrac{1}{2}(2\pm e^2)E(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots\ldots\ldots..(85).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R}=\frac{2}{2\pm e^2}\int_0^{\frac{1}{2}\pi}\frac{d\varphi}{\surd(1-e_1^2\sin^2\varphi)}=\frac{2}{2\pm e^2}F(e_1,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots..(86).$$

$$R_1=\tfrac{1}{2}(2\pm e^2)[(1-e_1^2\sin^2 2\theta)(1-e_1^2\cos^2 2\theta)]^{\frac{1}{2}}$$

$$=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)[1-e_2^2\sin^2(\tfrac{1}{2}\pi-4\theta)]^{\frac{1}{2}}.$$

Let $(\tfrac{1}{2}\pi-4\theta)=\lambda$. $\therefore\ d\theta=-\tfrac{1}{4}d\lambda$.

$$\therefore \int_0^{\frac{1}{2}\pi}R_1\,d\theta=\tfrac{1}{4}\int_{-\frac{1}{2}3\pi}^{\frac{1}{2}\pi}R_1\,d\lambda=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda+\tfrac{1}{4}\int_0^{\frac{1}{2}3\pi}R_1\,d\lambda=\tfrac{1}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda$$

$$+\tfrac{3}{4}\int_0^{\frac{1}{2}\pi}R_1\,d\lambda=\int_0^{\frac{1}{2}\pi}R_1\,d\lambda=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)\int_0^{\frac{1}{2}\pi}\sqrt{(1-e_2^2\sin^2\lambda)}d\lambda$$

$$=\tfrac{1}{4}(2\pm e^2)(2-e_1^2)E(e_2,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots\ldots\ldots(87).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R_1}=4\int_0^{\frac{1}{2}\pi}\frac{d\lambda}{R_1}=\frac{4}{(2\pm e^2)(2-e_1^2)}F(e_2,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots\ldots\ldots(88).$$

Similarly, $\displaystyle\int_0^{\frac{1}{2}\pi}R_2\,d\theta=\tfrac{1}{8}(2\pm e^2)(2-e_1^2)(2-e_2^2)E(e_3,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots(89).$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R_2}=\frac{8}{(2\pm e^2)(2-e_1^2)(2-e_2^2)}F(e_3,\ \tfrac{1}{2}\pi)\ \ldots\ldots\ldots\ldots\ldots\ldots(90).$$

$$\int_0^{\frac{1}{2}\pi}R_3\,d\theta=\frac{1}{2^4}(2\pm e^2)(2-e_1^2)(2-e_2^2)(2-e_3^2)E(e_4,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots(91).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R_3}=\frac{2^4}{(2\pm e^2)(2-e_1^2)(2-e_2^2)(2-e_3^2)}F(e_4,\ \tfrac{1}{2}\pi)\ldots\ldots\ldots\ldots(92).$$

. . . . . .

$$\int_0^{\frac{1}{2}\pi}R_n d\theta=\frac{1}{2^{n+1}}(2\pm e^2)(2-e_1^2)(2-e_2^2)\ldots(2-e_n^2)E(e_{n+1},\ \tfrac{1}{2}\pi)\ldots\ldots(93).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{R_n}=\frac{2^{n+1}}{(2\pm e^2)(2-e_1^2)(2-e_2^2)\ldots(2-e_n^2)}F(e_{n+1},\ \tfrac{1}{2}\pi)\ldots\ldots\ldots(94).$$

$$\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}(2m+1)}d\theta.$$

Let $\tfrac{1}{2}\pi-\theta=\beta$.

$$\therefore \int_0^{\frac{1}{2}\pi}[1-e^2\sin^2(\tfrac{1}{2}\pi-\theta)]^{\frac{1}{2}(2m+1)}d\theta=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^{\frac{1}{2}(2m+1)}d\beta$$

$$=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^m\sqrt{(1-e^2\sin^2\beta)}d\beta=M,\ \text{suppose}.$$

Let $m=0$. $\therefore\ M=\displaystyle\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\beta)}d\beta=E(e,\ \tfrac{1}{2}\pi).$

Let $m=1$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)\sqrt{(1-e^2\sin^2\beta)}d\beta$.

From (2) and (8), $M=\frac{1}{3}[2(2-e^2)E(e,\ \frac{1}{2}\pi)-(1-e^2)F(e,\ \frac{1}{2}\pi)]$........(95).

Let $m=2$. $\therefore M=\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\beta)^2\sqrt{(1-e^2\sin^2\beta)}d\beta$.

From (2), (8), and (15),

$M=\frac{1}{15}[(23-23e^2-8e^4)E(e,\ \frac{1}{2}\pi)-4(2-3e^2+e^4)F(e,\ \frac{1}{2}\pi)]$ ....(96),

and so on for other values of m.

To integrate $\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}$, let $\tan\delta=\frac{1}{\sqrt{(1-e^2)}}\tan\theta$.

$$\therefore \int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\sin^2\delta)^{\frac{1}{2}(2m-1)}d\delta,$$

a form like preceding. When $m=0$ we get

$$\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\cos^2\theta)}d\theta=\int_0^{\frac{1}{2}\pi}\sqrt{(1-e^2\sin^2\theta)}d\theta=E(e,\ \tfrac{1}{2}\pi).$$

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\cos^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=F(e,\ \tfrac{1}{2}\pi).$$

By letting $\cot\gamma=\frac{1}{\sqrt{(1-e^2)}}\cot\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2m+1)}}=\frac{1}{(1-e^2)^m}\int_0^{\frac{1}{2}\pi}(1-e^2\cos^2\gamma)^{\frac{1}{2}(2m-1)}d\gamma,\text{ a previous form.}$$

Letting $\tan^2\delta=\frac{1}{1-e^2}\tan^2\theta$, we get

$$\int_0^{\frac{1}{2}\pi}\frac{d\theta}{(1-c\cos^2\theta)(1-e^2\cos^2\theta)^{\frac{1}{2}(2m+1)}}$$

$$=\frac{1}{(1-c)(1-e^2)^m}\int_0^{\frac{1}{2}\pi}\frac{(1-e^2\sin^2\delta)^{m+1}d\delta}{(1+\frac{c-e^2}{1-c}\sin^2\delta)\sqrt{(1-e^2\sin^2\delta)}}.$$

We will now proceed to more general work.

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^2\theta}=-\cot\theta\sqrt{(1-e^2\sin^2\theta)}-e^2\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=(1-e^2)F(e,\ \theta)-E(e,\ \theta)-\cot\theta\sqrt{(1-e^2\sin^2\theta)}=u_1\ldots.(97).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta}=\tan\theta\sqrt{(1-e^2\sin^2\theta)}+e^2\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=F(e,\ \theta)-E(e,\ \theta)+\tan\theta\sqrt{(1-e^2\sin^2\theta)}=R_1\ \ldots\ldots\ldots\ldots\ldots\ (98).$$

$$\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^2\theta}$$

$$=F(e,\ \theta)-E(e,\ \theta)-\cot\theta\sqrt{(1-e^2\sin^2\theta)}=V_1\ \ldots\ldots\ldots\ldots\ldots\ (99).$$

$$\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^2\theta}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=F(e,\ \theta)-\frac{1}{1-e^2}E(e,\ \theta+\frac{\tan\theta}{1-e^2}\sqrt{(1-e^2\sin^2\theta)}=S_1\ \ \ldots\ \ldots.(100).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta}=-\frac{\cos\theta}{3}\left(\frac{1}{\sin^3\theta}+\frac{2}{\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)}+\frac{e^2}{3}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{e^2}{3}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{2e^2}{3}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{2(1-e^2)}{3}F(e,\ \theta)-$$

$$\frac{(2-e^2)}{3}E(e,\ \theta)-\frac{(2-e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=U_2\ ..(101).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta}=\frac{\sin\theta}{3}\left(\frac{1}{\cos^3\theta}+\frac{2}{\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{e^2}{3}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^3}{3}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{2e^2}{3}\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\tfrac{2}{3}F(e,\ \theta)-\frac{(2-e^2)}{3(1-e^2)}E(e,\ \theta)+\frac{(2-e^2)}{3(1-e^2)}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{3\cos^3\theta}\sqrt{(1-e^3\sin^2\theta)}=R_2\ \ \ldots\ldots\ldots\ldots\ldots\ (102).$$

$$\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$=\frac{(2+e^2)}{3}F(e,\ \theta)-\frac{2(1+e^2)}{3}E(e,\ \theta)-\frac{2(1+e^2)}{3}\cot\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{\cos\theta}{3\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=V_2\ \ \ldots\ldots\ldots\ldots\ldots\ (103).$$

$$\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}\,d\theta}{\cos^4\theta}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$=\frac{(2-3e^2)}{3(1-e^2)}F(e,\ \theta)-\frac{2(1-2e^2)}{3(1-e^2)^2}E(e,\ \theta)+\frac{2(1-2e^2)}{3(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{3(1-e^2)\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_2 \quad \ldots\ldots\ldots\ldots(104).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}=-\frac{\cos\theta}{5}\left(\frac{1}{\sin^5\theta}+\frac{4}{3\sin^3\theta}+\frac{8}{3\sin\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{e^2}{5}\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{15}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{8e^2}{15}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-7e^2-e^4)}{15}F(e,\ \theta)-\frac{(8-3e^2-2e^4)}{15}E(e,\ \theta)$$

$$-\frac{(8-3e^2-2e^4)}{15}\cot\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{(4-e^2)}{15}\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}=U_3 \quad \ldots\ldots\ldots\ldots\ldots(105).$$

$$\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}=\frac{\sin\theta}{5}\left(\frac{1}{\cos^5\theta}+\frac{4}{3\cos^3\theta}+\frac{8}{3\cos\theta}\right)\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{e^2}{5}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\frac{e^2}{15}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{4e^2}{15}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}$$

$$+\frac{8e^2}{15}\int\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-9e^2)}{15(1-e^2)}F(e,\ \theta)-\frac{(8-13e^2+3e^4)}{15(1-e^2)^2}E(e,\ \theta)$$

$$+\frac{8-13e^2+3e^4}{15(1-e^2)^2}\tan\theta\sqrt{(1-e^2\sin^2\theta)}+\frac{4-3e^2}{15(1-e^2)}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}=R_3 \quad \ldots\ldots\ldots\ldots\ldots\ldots(106).$$

$$\int\frac{d\theta}{\sin^6\theta\sqrt{(1-e^2\sin^2\theta)}}=e^2\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}+\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\sin^6\theta}$$

$$=\frac{(8+3e^2+4e^4)}{15}F(e,\ \theta)-\frac{(8+7e^2+8e^4)}{15}E(e,\ \theta)-\frac{\cos\theta}{5\sin^5\theta}\sqrt{(1-e^2\sin^2\theta)}$$

$$-\frac{(8+7e^2+8e^4)}{15}\tan\theta\sqrt{(1-e^2\sin^2\theta)}-\frac{4(1+e^2)}{15}\cdot\frac{\cos\theta}{\sin^3\theta}\sqrt{(1-e^2\sin^2\theta)}=V_3 \quad ..(107).$$

[To be Concluded.]

GAUSS AND THE NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED.

Carl Friedrich Gauss Werke, Band VII. Goettingen. 1900. 4to. Pp. 458.

We are so accustomed to the German professor who does, we hardly expect the German professor who does not. Such, however, was Schering of Goettingen, who so long held possession of the papers left by Gauss.

Schering had planned and promised to publish a supplementary volume, but never did, and only left behind him at his death certain preparatory attempts thereto, consisting chiefly of excerpts copied from the manuscripts and letters left by Gauss. Meantime these papers for all these years were kept secret and even the learned denied all access to them.

Schering dead, his work has been quickly and ably done, and here we have a stately quarto of matter supplemental to the first three volumes, and to the fourth volume with the exception of the geodetic part.

Of chief interest for us is the geometric portion, pages 159—452, edited by just the right man, Professor Staeckel of Kiel.

One of the very greatest discoveries in mathematics since ever the world began is, beyond peradventure, the non-Euclidean geometry.

By whom was this given to the world in print? By a Hungarian, John Bolyai, who made the discovery in 1823, and by a Russian, Lobachévski, who had made the discovery by 1826. Were either of these men prompted, helped, or incited by Gauss, or by any suggestion emanating from Gauss? No, quite the contrary.

Our warrant for saying this with final and overwhelming authority is this very eighth volume of Gauss's works, just now at last put in evidence, published to the world.

The geometric part opens, page 159, with Gauss's letter of 1799 to Bolyai Farkas the father of John (Bolyai János), which I gave years ago in my Bolyai as demonstrative evidence that in 1799 Gauss was still trying to prove Euclid's the only non-contradictory system of geometry, and also the system of objective space. The first is false; the second can never be proven.

But both these friends kept right on working away at this impossibility, and the more hot-headed of the two, Farkas, finally thought he had succeeded with it, and in 1804 sent to Gauss his "Goettingen Theory of Parallels." Gauss's judgment on this is the next thing given (pages 160—2). He shows the weak spot. "Could you *prove*, that $dkc = ckf = fkg$, etc., then were the thing perfect. However, this theorem is indeed true, only difficult, with already presupposing the theory of parallels, to prove rigorously." Thus in 1804 instead of having or giving any light, Gauss throws his friend into despair by intimating that the link missing in his labored attempt is true enough but difficult to prove without *petitio principii*.

Of course we now know it is *impossible* to prove. Anything is impossible to prove which is the equivalent of the parallel-postulate. Yet both the friends continue their strivings after this impossibility.

In this very letter Gauss says: "I have indeed yet ever the hope, that those rocks sometime, and indeed before my end, will allow a through passage."

Farkas in 1808, December 27, writes to Gauss: "Oft thought I, gladly would I, as Jacob for Rachel serve, in order to know the parallels founded, even if by another.

Now just as I thought it out on Christmas night, while the Catholics were celebrating the birth of the Saviour in the neighboring church, yesterday wrote it down, I send it to you enclosed herewith. To-morrow must I journey out to my land, have no time to revise, neglect I it now, may be a year is lost, or indeed find I the fault, and send it not, as has already happened with hundreds, which I as I found them took for genuine. Yet it did not come to writing those down, probably because they were too long, too difficult, too artificial, but the present I wrote off at once. As soon as you can, write me your real judgment."

This letter Gauss never answered, and never wrote again until 1832, a quarter of a century later, when the non-Euclidean geometry had been published by both Lobachévski and Bolyai János.

This settles now forever all question of Gauss having been of the slightest or remotest help or aid to young János, who in 1823 announced to his father Farkas in a letter still extant, which I saw at the Reformed College in Maros-Vásárhely, where Farkas was professor of mathematics, his discovery of the non-Euclidean geometry as something undreamed of in the world before.

This immortal letter, a charming and glorious outpouring of pure young genius, speaks as follows:

"Temesvár, 3 Nov. 1823.

My dear and good father. I have so much to write of my new creations, that it is at the moment impossible for me to enter into great detail, so I write you only on a quarter of a sheet. I await your answer to my letter of two sheets; and perhaps I would not have written you before receiving it, if I had not wished to address to you the letter I am writing to the Baroness, which letter I pray you to send her.

First of all I reply to you in regard to the binomial.

*   *   *   *   *   *   *   *   *

Now to something else, so far as space permits. I intend to write, as soon as I have put it into order, and when possible to publish, a work on parallels. At this moment it is not yet finished, but the way I have hit upon promises me with certainty the attainment of the goal, if it in general is attainable. It is not yet attained, but I have discovered such magnificent things that I am myself astonished at them. It would be damage eternal if they were lost. When you see them, my father, you yourself will acknowledge it. Now I cannot say more of them, only so much: *that from nothing I have created a wholly new world.* All that I have hitherto sent you compares to this only as a house of cards to a castle.

P. S. I dare to judge absolutely and with conviction of these works of my spirit before you, my father; I do not fear from you any false interpretation (that certainly I would not merit), which signifies that, in certain regards, I consider you as a second self."

In his Autobiography János says: "First in the year 1823 did I completely penetrate through the problem according to its essential nature, though also afterward further completions came thereto. I communicated in the year 1825 to my former teacher, Herrn Johann Walter von Eckwehr (later imperial-royal general), a written paper, which is still in his hands. On the prompting of my father I translated my paper into Latin, in which it appeared as *Appendix* to the *Tentamen* in 1832."

So much for Bolyai.

The equally complete freedom of Lobachévski from the slightest idea that Gauss had ever meditated anything different from the rest of the world on the parallels I showed in "Science," Vol. IX, No. 232, page 813—817.

Passing on to the next section, pages 163-4, in the new volume of Gauss, we find it important as showing that in 1805 Gauss was still a baby on this subject. It is an erroneous pseudo-proof of the impossibility of what in 1733 Saccheri had called "hypothesis anguli obtusi." To be sure Saccheri himself thought he had proven this hypothesis inadmissable, so that Gauss blundered in good company; but his pupil Riemann in 1854 showed that this hypothesis gives a beautiful non-Euclidean geometry, a new universal space, now justly called the space of Riemann.

Passing on, we find that in 1808, Schumacher writes: "Gauss has led back the theory of parallels to this, that if the accepted theory were not true, there must be a constant *a priori* line given in length, which is absurd. Yet he himself considers this work still not conclusive."

Again, with the date April 27, 1813, we read: "In the theory of parallels we are even now not farther than Euclid was. This is the *partie honteuse* (shameful part) of mathematics, which soon or late must receive a wholly different form."

Thus in 1813 there is still no light.

In April, 1816, Wachter on a visit to Goettingen had a conversation with Gauss whose subject was what he calls the anti-Euclidean geometry. On December 12, 1816, he writes to Gauss a letter which shows that this anti-Euclidean geometry, as he understands it, far from being the non-Euclidean geometry of Lobachévski and Bolyai János, was a monstrous conglomerate blunder.

The letter as here given by Staeckel, pages 175—176, is as follows:

* * * "Consequently the anti-Euclidean or your geometry would be true. However the *constant* in it remains undetermined: why? may perhaps be made comprehensible by the following. * * * The result of the foregoing may consequently be so expressed:

The Euclidean geometry is false; but nevertheless the true geometry must begin with the same eleventh Euclidean axiom or with the assumption of

lines and surfaces which have the property presumed in that axiom. Only instead of the straight line and plane are to be put the great circle of that sphere described with infinite radius together with its surface. From this comes indeed the one inconvenience, that the parts of this surface are merely symmetric, not, as with the plane, congruent; or that the radius out on the one side is infinite, on the other imaginary. Only it is clear how that inconvenience is again overbalanced by many other advantages which the construction on a spherical surface offers : *so that probably also then even*, if the Euclidean geometry were true, the necessity no longer indeed exists, to consider the plane as an infinite spherical surface, though still the fruitfulness of this view might recommend it.

Only, as I thought through all this, as I had already fully settled myself about the result, in part since I believed I had recognized the ground (la métaphysique) of that indeterminateness necessarily inherent in geometry,—also even the complete indecision in this matter, then, if that proof against the Euclidean geometry, as I could not expect, were not to be considered as stringent—; in part, while yet not to consider as lost all the many previous researches in plane geometry : but to be used with few modifications, and if still also the theorems of solid geometry and mechanics might have approximate validity, at least to a quite wide limit, which perhaps yet could be more nearly determined ; I found this evening—just while busied with an attempt to find an entrance to your transcendental trigonometry, and while I could not find in the plane sufficing, determinate functions thereto, going on to space constructions, to my no small delight the following *demonstration for the Euclidean parallel-theory.* * * * Just in the idea to conclude I remark still, that the above proof for the Euclidean parallel-theory is fallacious. * * * Consequently has here also the hope vanished, to come to a fully decided result, and I must content myself again with the above cited. Withal I believe I have made upon that way at least a step toward your transcendental trigonometry, since I, with aid of the spherical trigonometry, can give the ratios of all constants, at least *by construction of the right-angled triangle.* I yet lack the actual reckoning of the base of an isosceles triangle from the side, to which I will seek to go from the equilateral triangle."

If Gauss's transcendental trigonomety were as sad a hodge-podge as the anti-Euclidean geometry here explained by Wachter, it is fortunate that nothing was ever given about it but its name. *Requiescat in pace.*

Yet Gauss writes, April 28, 1817 :

"Wachter has printed a little piece on the foundations of geometry. Though Wachter has penetrated farther into the essence of the matter than his predecessors, yet is his proof not more valid than all others."

We come now to an immortal epoch, that of the discovery of the real non-Euclidean geometry by Schweikart, and his publication of it under the name of Astralgeometry.

On the twenty-fifth of January, 1819, Gerling writes to Gauss :

"Apropos of parallel-theory I must tell you something, and execute a commission. I learned last year, that my colleague Schweikart (prof. juris,

now Prorector) formerly occupied himself much with mathematics and particularly also had written on parallels. So I asked him to lend me his book. While he promised me this, he said to me, that now indeed he perceived how errors were present in his book (1808) (he had, for example, used quadrilaterals with equal angles as a primary idea), however that he had not ceased to occupy himself with the matter, and was now about convinced, that without some datum the Euclidean postulate could not be proved, also that it was not improbable to him, that our geometry is only a chapter of a more general geometry.

Then I told him how you some years ago had openly said, that since Euclid's time we had not in this really progressed; yes, that you had often told me, how you through manifold occupation with this matter had not attained to the proof of the absurdity of such a supposition.—Then when he sent me the book asked for, the enclosed paper accompanied it, and shortly after (end of December) he asked me orally, when convenient to enclose to you this paper of his, and to ask you in his name to let him know when convenient your judgment on these ideas of his.

The book itself has, apart from all else, the advantage that it contains a copious bibliography of the subject; which he also, as he tells me, has not ceased still further to add to."

Now comes, pages 180—181, the precious enclosure, dated Marburg, December, 1818, which, though so brief, may fairly be considered the first *published* [not printed] treatise on non-Euclidean geometry. It is a pleasure to give this here in English for the first time.

The non-Euclidean Geometry of 1818.

BY SCHWEIKART.

"There is a two-fold geometry,—a geometry in the narrower sense—the Euclidean; and an astral science of magnitude.

The triangles of the latter have the peculiarity, that the sum of the three angles is not equal to two right angles.

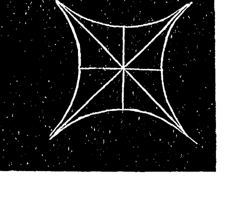
This presumed, it can be most rigorously proven:

(*a*) That the sum of the three angles in the triangle is *less* than two right angles;

(*b*) that this sum becomes ever smaller, the more content the triangle encloses;

(*c*) That the altitude of an isosceles right-angled triangle indeed ever increases, the more one lengthens the side, that it however cannot surpass a certain line, which I call the *constant*.

Squares have consequently the following form:



Is this constant *for us* half the earth's axis (as a consequence of which each line drawn in the universe from one fixed star to another, which are ninety degrees apart from one another, would be a tangent of the earth-sphere), so is it in relation to the spaces occuring in daily life infinitely great.

The Euclidean geometry holds good only under the presupposition, that

the constant is infinitely great. Only then is it true, that the three angles of every triangle are equal to two right angles ; also this can be easily proven if one takes as given the proposition, that the constant is infinitely great."

Such is the brief declaration of independence of this hero.

Nor was Schweikart's courage and independence without farther issue. Under his direct influence his own nephew Taurinus developed the real non-Euclidean trigonometry and published it in 1825 with successful application to a number of problems.

Moreover this teaching of Schweikart's made converts in high places. In the letter of Bessell to Gauss of February 10, 1829 (page 201) he says :

"Through that which Lambert said, and what Schweikart disclosed orally, it has become clear to me, that our geometry is incomplete, and should receive a correction, which is hypothetical and, if the sum of the angles of the plane triangle is equal to a hundred and eighty degrees, vanishes.

That were the *true* geometry, the Euclidean the *practical*, at least for figures on the earth."

The complete originality and independence of Schweikart and of Lobachévski is recognized as a matter of course in the correspondence between Gauss and Gerling, who writes, page 238, "The Russian steppes seem therefore indeed a proper soil for these speculations, for Schweikart (now in Koenigsberg) invented his 'Astral-Geometry' while he was in Charkow."

This fixes the date of the first conscious creation and naming of the non-Euclidean geometry as between 1812 and 1816.

Gauss adopts and uses for himself this first name, Astralgeometry [1832, page 226 ; 1841, page 232].

At length the true prince comes. On February 14, 1832, Gauss receives the profound treatise of the young Bolyai János, the most marvellous two dozen pages in the history of thought. Under the first impression Gauss writes privately to his pupil and friend Gerling of the ideas and results as "mit grosser eleganz entwickelt." He even says "I hold this young geométer von Bolyai to be a genius of the first magnitude."

Now was Gauss's chance to connect himself honorably with the non-Euclidean geometry, already independently discovered by Schweikart, by Lobachévski, by Bolyai János.

Of two utterly worthless theories of parallels Gauss had already given extended notices in the Goettingische gelehrte Anzeigen (this volume pages 170—174, and 183—185).

To this marvel of János, Gauss vouchsafed never one printed word.

As Staeckel gently remarks, this certainly contributed thereto, that the worth of this mathematical gem was first recognized when John had long since finished his earthly career.

The 15th of December, 1902, will be the centenary of the birth of Bolyai János. Should not the learned world endeavor to arouse the Magyars to honor Hungary by honoring then this truest genius her son ?

Austin, Texas.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the October Number.]

CHAPTER X.

146. Combinatory multiplication gives us a very neat solution of sets of simultaneous equations. Let n equations of the first degree containing n unknowns be given to find the value of the unknowns. Let the n equations be

$$\begin{aligned}
&\alpha_{11}x_1+\alpha_{21}x_2+\ldots\ldots+\alpha_{n1}x_n=\beta_1\\
&\alpha_{12}x_1+\alpha_{22}x_2+\ldots\ldots+\alpha_{n2}x_n=\beta_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\alpha_{1n}x_1+\alpha_{2n}x_2+\ldots\ldots+\alpha_{nn}x_n=\beta_n
\end{aligned}$$

We multiply the first of these equations through by e_1, the second through by e_2, and so on, and the last by e_n, (where $[e_1e_2\ldots\ldots e_n]=1$), and add the resulting equations. Then if

$$\begin{aligned}
&\alpha_{11}e_1+\alpha_{12}e_2+\ldots\ldots+\alpha_{1n}e_n=a_1\\
&\alpha_{21}e_1+\alpha_{22}e_2+\ldots\ldots+\alpha_{2n}e_n=a_2\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
&\alpha_{n1}e_1+\alpha_{n2}e_2+\ldots\ldots+\alpha_{nn}e_n=a_n\\
&\beta_1e_1+\beta_2e_2+\ldots\ldots\ldots+\beta_ne_n=b,
\end{aligned}$$

we have for the sum of the products referred to above, the equation

$$x_1a_1+x_2a_2+\ldots\ldots+x_na_n=b.$$

By 20 this one equation replaces or transforms the set originally given us. Now in order to find x_1, multiply the last equation by $[a_2a_3\ldots\ldots a_n]$. This gives (52, 43)

$$x_1[a_1a_2\ldots\ldots a_n]=[ba_2a_3\ldots\ldots a_n]$$

$$\text{or, } x=\frac{[ba_2a_3\ldots\ldots a_n]}{[a_1a_2\ldots\ldots a_n]}$$

Replacing the a's and b by their values, by 45 we have the usual expression for x_1 in terms of the coefficients.

CHAPTER XI.

APPLICATION TO TRIGONOMETRY.

147. DEFINITION.—The angle between two quantities is that angle ($<\pi$) whose cosine equals the inner product of the two quantities divided by the product of their scalar coefficients. Thus

$$\cos\angle ab=[a\mid b]\div\alpha\beta$$

where a and b are two quantities and α and β are their numerical values (123).

Again, if $a, b, c\ldots.$ are quantities of the first order, $\alpha, \beta, \gamma\ldots.$ are their respective numerical values. $\sin(a\ b\ c\ldots.)$ is that numerical quantity which equals

$$\frac{[a\ b\ c\ldots.]}{\alpha\beta\gamma\ \ldots}$$

and is not negative. Thus (123) $\sin^2(abc)=\frac{[a\ b\ c\ldots.]^2}{\alpha^2\beta^2\gamma^3\ldots.}$.

148. *If a and b are quantities of the first order*, $\sin(ab)=\sin\angle ab$.

PROOF.— $\sin^2(ab)=\frac{[a\,b]^2}{\alpha^2\beta^2}=\frac{a^2b^2-[a\mid b]^2}{\alpha^2\beta^2}$ (144)

$$=\frac{\alpha^2\beta^2-[a\mid b]^2}{\alpha^2\beta^2}\ (123)=1-\frac{[a\mid b]^2}{\alpha^2\beta^2}\ (123)$$

$$=1-\cos^2\angle ab=\sin^2\angle ab\ (147).$$

Then if $\sin(ab)$ is never negative and $\angle ab<\pi$, $\sin(ab)=\sin\angle ab$.

149. *If $\alpha, \beta, \gamma, \delta$ are the numerical values of a, b, c, d, by* 147 *and* 148,

$$[ab\mid cd]=\alpha\beta\gamma\delta\sin\angle ab\sin\angle cd\cos\angle(ab.ed).$$

150. *The normal projection of A on a quantity B of the same order is numerically equal to $A\cos\angle AB$.*

PROOF.—If A' is the normal projection of A on B (134)

$$A'=\frac{[A\mid B]B}{\beta^2}=\frac{\alpha\beta\cos\angle AB.B}{\beta^2}\ (119,\ 144)$$

$$=\alpha\cos\angle AB\,.\,\frac{B}{\beta}=\text{(numerically) } A\cos\angle AB.$$

151. The two expressions $[a\mid b]$ and $[ab]$, where a and b are vectors, play a very important part in mathematics. They occur yoked together in quaternions and apart in the *Ausdehnungslehre*, typifying the two products, the inner and outer. Numerically, as we have just seen, $[a\mid b]$ is the *projection* of either vector on the other multiplied by the tensor of the other ; $[ab]$, on the other

hand, is the area of the parallelogram whose adjacent sides are a and b, or, when the tensor of one vector is unity, it is equal numerically to the *perpendicular* from the extremity of the other vector on the first, when they go out from the same origin, or, when both tensors are unity, it is equal numerically to the sine of the angle between the given vectors (148).

152. *If a, b, c, ... are normal to each other and k is any quantity numerically derived from them, we have*

$$\frac{k}{\varkappa}=\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.$$

PROOF.—Let $k=xa+yb+\ldots.$ Then to find x multiply each member by $|a$. There results, since $[b\,|\,a]=0$, etc., $[k\,|\,a]=x[a\,|\,a]$. Finding the value of $y\ldots.$ in the same way and substituting we get the equation as given above.

153. If a, b, $c\ldots.$ are normal to one another and k and l are two quantities numerically derivable from a, b, ...

$$\cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

PROOF.—From 147, we have

$$\cos\angle kl=\frac{[k\,|\,l]}{\varkappa\lambda}=\left[\frac{k}{\varkappa}\Big|\frac{l}{\lambda}\right]\equiv$$

$$[(\frac{a}{\alpha}\cos\angle ak+\frac{b}{\beta}\cos\angle bk+\ldots.)\,|\,(\frac{a}{\alpha}\cos\angle al+\frac{b}{\beta}\cos\angle bl+\ldots.)]\ (152)$$

$$=\frac{a^2}{\alpha^2}\cos\angle ak\cos\angle al+\frac{b^2}{\beta^2}\cos\angle bk\cos\angle bl+\ldots.$$

$$\therefore\ \cos\angle kl=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

154. If a, b, c, ... are normal to each other and k is numerically derivable from them, we have by putting $l=k$ in 153

$$1=\cos^2\angle ak+\cos^2\angle bk+\ldots.$$

155. If a, b, c, ... are normal to each other, and k and l two quantities numerically derivable from them are normal to each other, 153 gives

$$0=\cos\angle ak\cos\angle al+\cos\angle bk\cos\angle bl+\ldots.$$

156. Writing in the formula of 144 a for p_1, b for p_2, c for q_1, d for q_2 gives

$$\sin\angle ab\sin\angle cd\cos\angle(ab.cd)=\cos\angle ac\cos\angle bd-\cos\angle bc\cos\angle ad.$$

In this formula if c and d are replaced respectively by a and c there results

$$\sin\angle ab \sin\angle ac \cos\angle(ab.ac)=\cos\angle bc-\cos\angle ba\cos\angle ac,$$

a familiar formula of spherical trigonometry.

157. The last formula of 145, by substituting a, b, c for p_1, p_2, p_3, and a, b, c for q_1, q_2, q_3 gives (147)

$$\sin^2(abc)=1-\cos^2\angle bc-\cos^2\angle ca-\cos^2\angle ab+2\cos\angle bc\cos\angle ca\cos\angle ab.$$

158. *The formula* $(a+b)^2=a^2+2[a \mid b]+b^2=\alpha^2+2\alpha\beta\cos\angle ab+b^2$ gives the familiar extension of the Pythagorean proposition.

159. Let a, b, c be plane segments whose sum is the fourth face of a tetraedron of which they are the other three (75). Then

$$\begin{aligned}(a+b+c)^2&=a^2+b^2+c^2+2[b \mid c]+2[c \mid a]+2[a \mid b]\\&=\alpha^2+\beta^2+\gamma^2+2\beta\gamma\cos\angle bc+2\alpha\gamma\cos\angle ca+2\alpha\beta\cos\angle ab,\end{aligned}$$

which is the extension of the preceding result to space.

In words:—The square of the base of any tetraedron is equal to the sum of the squares of the lateral faces dimished by twice the products of each pair of lateral faces times the cosine of the diedral angle between them.

CHAPTER XII.

APPLICATION TO ANALYTIC GEOMETRY.

160. Let p_1, p_2, p_3 represent three unit points, and suppose their product is unity (57). Then

$$[p_1p_2p_3]=[p_1(p_2p_3+p_3p_1+p_1p_2)]=1. \text{ (43)}$$

But if p denote any other unit point in the plane of $[p_1p_2p_3]$, by 94 we may replace p_1 by p in this product, getting

$$[p(p_2p_3+p_3p_1+p_1p_2)]=[p \mid (p_1+p_2+p_3)]=1. \text{ (57, 58)}$$

Let p_s denote a unit point which is the mean of the reference points. Then $p_1+p_2+p_3=3p_s$ (81). Substituting this value of $p_1+p_2+p_3$ in the equation above, we have

$$3[p \mid p_s]=1, \text{ or in solid space, } 4[p \mid p_s]=1.$$

161. *The equation* $p=xp_1+yp_2+zp_3$ *represents a straight line, provided* x, y, z *satisfy a linear equation, as* $ax+by+cz=0$.

To see this let us eliminate z. Then

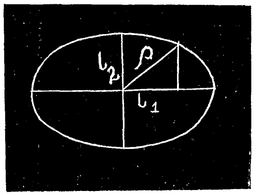
$$p=\frac{1}{c}\{x(cp_1-ap_3)+y(cp_2-bp_3)\}.$$

Thus, by 80, p lies on the right line through cp_1-ap_3 and cp_2-bp_3.

162. The equation $[p\,p_1p_2]=0$, in which p_1 and p_2 are constants and p is a variable is the equation of a straight line (94).

The equation $[pL]=0$, *where* p *is a point and* L *a line is the point equation of a straight line if* p *is variable and* L *is constant, and the line equation of the point* p *if* L *is variable and* p *is constant.*

163. *The Cartesian equations of the central conics, the ellipse and the hyperbola in the inner product notation* (151) *are*



$$\left(\frac{\rho\,|\,\iota_1}{a}\right)^2 \pm \left(\frac{\rho\,|\,\iota_2}{b}\right)^2=1,$$

where ι_1 and ι_2 are unit vectors along the major and minor axes and ρ is the radius vector from the center to any point. Suppose we set

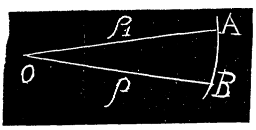
$$\frac{\rho\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho\,|\,\iota_2}{b^2}\iota_2 \equiv \phi\rho.$$

Then the equation for the central conics reduces to

$$\rho\,|\,\phi\rho=1.$$

164. Differentiation.—Let ρ be a radius vector from an origin O to a curve AB. Then if ρ be made to approach indefinitely close to ρ_1, we have

$$\text{Limit}\frac{\rho-\rho_1}{AB}=\frac{d\rho}{ds}=a\ unit\ vector$$



in the direction of the tangent at A. This is taken to be the meaning of $\frac{d\rho}{ds}$ no matter whether ρ be a *vector* from the origin O, or a *point* moving from B to A on the curve AB.

165. The function ϕ (163) possesses the property that $\rho\,|\,\phi\rho_1=\rho_1\,|\,\phi\rho$. Thus

$$\rho\,|\,\phi\rho_1\equiv\rho\,|\left(\frac{\rho_1\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho_1\,|\,\iota_2}{b^2}\iota_2\right)=\rho_1\,|\left(\frac{\rho\,|\,\iota_1}{a^2}\iota_1 \pm \frac{\rho\,|\,\iota_2}{b^2}\iota_2\right).$$

166. Differentiating the equation $\rho\,|\,\phi\rho=1$ (163), we get

$$d\rho\,|\,\phi\rho+\rho\,|\,\phi d\rho=2d\rho\,|\,\phi\rho=0. \quad (165)$$

Now if $d\rho$ is parallel to the tangent at the extremity of ρ, $\phi\rho$ is parallel to the normal (124).

If ρ_t and ρ_n be vectors to any point of the tangent and normal, respectively, and ρ_1 that to the point of contact, the equation of the tangent may be written $[(\rho_t-\rho_1)\,|\,\phi\rho_1]=0$, or $[\rho_t\,|\,\phi\rho_1]=1$, and that of the normal $[(\rho_n-\rho_1)\phi\rho_1]=0$.

The ρ's may also be thought of as representing points.

167. Let p_1, p_2, p_3 denote the vertices of a reference triangle whose sides are of unit length and p any point in their plane. Then $|p_1=p_2p_3$, $|p_2=p_3p_1$, $|p_3=p_1p_2$, and $p|p_1$, $p|p_2$, $p|p_3$ are proportional to the perpendiculars from p on the several sides of the triangle (94).

We shall consider only homogeneous equations. For, if any equation should not be homogeneous in p, all that is necessary to make it such is to introduce the factor $1=3p|p_s$ (160). Now the most general form of the equation of the second degree in trilinear coördinates is

$$a[p|p_1]^2+b[p|p_2]^2+c[p|p_3]^2+2d[p|p_2][p|p_3]$$
$$+2e[p|p_3][p|p_1]+2f[p|p_1][p|p_2]=0.$$

Let $[(ap_1+fp_2+ep_3)p|p_1]+[(fp_1+bp_2+dp_3)p|p_2]$
$$+[(ep_1+dp_2+cp_3)p|p_3]\equiv\phi p.$$

When this value of ϕp is substituted in the preceding equation it reduces to $p|\phi p=0$. Hence $p|\phi p=0$ is the equation for all quadric curves whether central or non-central. Had quadriplaner coördinates been employed and the corresponding expressions constructed, an equation would have resulted representing any and all quadric surfaces. The same method may be used in getting the equation of the quadric in n-dimensional space.

REMARK.—The introduction of the ϕ function from Hamilton into the *Ausdehnungslehre* is due to Professor Hyde. (*Directional Calculus*, page 103). He shows that point analysis gives a means of changing the ordinary Cartesian equations into equations analogous to those of trilinear coördinates and then of generalizing the application of the equation $p|\phi p=0$ to include the case of quadrics, central and non-central.

[To be Concluded.]

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

132. Proposed by WILLIAM SYMMONDS, A.M., Professor of Mathematics, Santa Rosa College, Sebastopol, Cal.

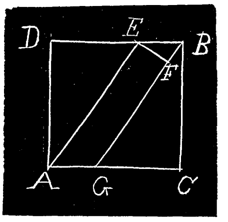
A road 60 feet wide crosses a square acre of land. The west line of the road passes through the southwest corner of the land, while the east line of the former passes through the northeast corner of the latter. What fraction of the land is included in the road?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; D. G. DORRANCE, Jr., Camden, N. Y.; J. SCHEFFER, A. M., Hagerstown, Md.; and G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Let $ACBD$=the square acre, $AGBE$=the road, and EF perpendicular to BG, =60 feet.

Put $a=AC=BC$, and $b=EF$.

Let $x=BE$, and $y=BG$.



Then $a-x=GC$, ax=area of road, and x/a=fractional part of square acre included in the road.

From the similar right triangles EFB and BCG, we have $x:y=b:a$; whence $y=ax/b$.

Also from right triangle BCG, $y=\sqrt{[a^2+(a-x)^2]}$.

$\therefore \dfrac{a^2x^2}{b^2}=2a^2-2ax+x^2$.

Whence $x=\dfrac{ab[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$, and $\dfrac{x}{a}=\dfrac{b[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$.

Substituting the numerical values, $b=60$ and $a^2=43560$ [=the area of an acre in square feet], we obtain $x/a=.344$.

QUERY. When $a=b$, what is the value of $\dfrac{ab[-b\pm\sqrt{(2a^2-b^2)}]}{a^2-b^2}$? GRUBER.

ANSWER. By differentiating both numerator and denominator with respect to b and then reducing we find the value of the expression to be equal to a, for either the + or − sign. It may also be shown as follows:

$$\frac{ab[\pm\sqrt{(2a^2-b^2)}-b]}{a^2-b^2}=\frac{ab[\pm\sqrt{(2a^2-b^2)}-b]}{\frac{1}{2}(2a^2-b^2-b^2)}$$

$$=\frac{2ab[\pm\sqrt{(2a^2-b^2)}-b]}{[\sqrt{(2a^2-b^2)}+b]\,[\sqrt{(2a^2-b^2)}-b]}=\frac{2ab}{\sqrt{(2a^2-b^2)}+b}\text{ or }-\frac{2ab}{\sqrt{(2a^2-b^2)}-b}$$

$$=a\text{ or }-\infty.$$

These values might have been found by making the assumption that $a=b$ in the equation from which the expression arose.

If, however, we write the denominator of the expression for the roots $2a^2-b^2-a^2=[\sqrt{(2a^2-b^2)}+a][\sqrt{(2a^2-b^2)}-a]$, and then divide $+\sqrt{(2a^2-b^2)}-b$ by $[\sqrt{(2a^2-b^2)}-a]$ we get

$$1+\frac{a-b}{\sqrt{(2a^2-b^2)}}+\frac{a(a-b)}{(2a^2-b^2)}+\frac{a^2(a-b)}{(2a^2-b^2)^{\frac{3}{2}}}+\text{etc.},$$

and the value of the root is

$$\frac{ab}{[\sqrt{(2a^2-b^2)}+a]}\left(1+\frac{a-b}{(2a^2-b^2)^{\frac{1}{2}}}+\frac{a(a-b)}{(2a^2-b^2)}+\frac{a^2(a-b)}{(2a^2-b^2)^{\frac{3}{2}}}+\text{etc.}\right)$$

While each of these terms after the first, approach 0 as $a \doteq b$, making it appear that from this view one root is $\frac{1}{2}a$ instead of a, as found above; yet by writing the series as follows:

$$1+(a-b)\left[\frac{a}{\sqrt{(2a^2-b^2)}}+\frac{a^2}{(2a^2-b^2)}+\frac{a^3}{(2a^2-b^2)^{\frac{3}{2}}}+\text{ etc.}\right],$$

it is seen that this factor takes the form, in the limit, $1+0\times\infty$; and, therefore, this method is no more capable of yielding a determinate result than is the original expression. EDITOR F.

ALGEBRA.

108. Proposed by GEORGE LILLEY, Ph.D., LL.D., Professor of Mathematics, State University, Eugene, Or.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent. annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the endorsement on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$.

[This problem is the same as No. 86, Miscellaneous. See the solutions in that department.]

109. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

$\dfrac{1}{\sqrt{(x+1)}}+\dfrac{1}{\sqrt{(x-1)}}=\dfrac{1}{\sqrt{(x^2-1)}}$; find value of x satisfying the equation.

Solution by the PROPOSER.

This problem was proposed for the purpose of explaining the singular fact (singular to those who do not possess more than a mechanical knowledge of algebra) that the value of the unknown does not satisfy the original equation.

By transposing and factoring, the original equation may be written

$$\frac{1}{\sqrt{(x^2-1)}}[\sqrt{(x-1)}+\sqrt{(x+1)}-1]=0,$$

which is equivalent to the system of equations,

$$\left\{\begin{array}{ll}\dfrac{1}{\sqrt{(x^2-1)}}=0 \ldots\ldots\ldots\ldots\ldots\ldots & A. \\ \sqrt{(x-1)}+\sqrt{(x+1)}-1=0 \ldots\ldots\ldots & B.\end{array}\right\}$$

The solution of A gives $x=\infty$, which value satisfies the original equation.

From B we have $\sqrt{(x+1)}=1-\sqrt{(x-1)}$. Squaring both sides, transposing and combining, we have $1=-2\sqrt{(x-1)}$.

From this last, by squaring, we obtain $1=4(x-1)$, from which we find,

$x=\frac{5}{4}$. This value of x does not satisfy B, neither does it satisfy the original equation.

By studying the two operations of squaring, in the solution of B, it will be seen that extraneous equations were introduced. The squaring the first time was equivalent to multiplying B by $\sqrt{(x-1)}+\sqrt{(x+1)}+1$, that is, the equation resulting from squaring the first time is equivalent to

$$[\sqrt{(x-1)}+\sqrt{(x+1)}+1][\sqrt{(x-1)}+\sqrt{(x+1)}-1]=0\ldots.(1),$$

and the second operation of squaring which gives the equation $1=4(x-1)$ is equivalent to multiplying (1) by $[\sqrt{(x-1)}-\sqrt{(x+1)}+1][\sqrt{(x-1)}-\sqrt{(x+1)}-1]$, that is, the equation $1=4(x-1)$ is equivalent to the equation

$$[\sqrt{(x-1)}+\sqrt{(x+1)}+1][\sqrt{(x-1)}+\sqrt{(x+1)}-1]$$
$$[\sqrt{(x-1)}-\sqrt{(x+1)}+1][\sqrt{(x-1)}-\sqrt{(x+1)}-1]=0.$$

This equation, therefore, is equivalent to the system of equations

$$\left\{\begin{array}{l}\sqrt{(x-1)}+\sqrt{(x+1)}+1=0=P \\ \sqrt{(x-1)}+\sqrt{(x+1)}-1=0=Q \\ \sqrt{(x-1)}-\sqrt{(x+1)}+1=0=R \\ \sqrt{(x-1)}-\sqrt{(x+1)}-1=0=S\end{array}\right\}$$

Of these, the only one that is satisfied by the value $x=\frac{5}{4}$ is R. The solution, therefore, of any one of these equations gives the same value of x, viz., $x=\frac{5}{4}$, and this value of x satisfies R only.

This whole subject of *derivation of equations* has been neglected by most writers on algebra in America until quite recently. The recent texts of Fisher and Schwatt, Beman and Smith, and several others have given considerable attention to the subject.

Professor Chrystal has given a very good treatment of the subject in his Algebra, Vol. I, § XIV. On page 285, he says, "There are few parts of algebra more important than the logic of the derivation of equations, and few, unhappily, that are treated in a more slovenly fashion in elementary teaching."

This problem was also solved in a very excellent manner by *H. C. WHITAKER*, *G. B. M. ZERR*, *COOPER D. SCHMITT*. *W. H. CARTER*, *W. W. LANDIS*, *CHARLES C. CROSS*, *J. M. BOORMAN*, *J. D. CRAIG*, *A. F. KOVARIK*, *ELMER SCHUYLER*, and *J. SCHEFFER*.

110. Proposed by J. C. CORBIN, Pine Bluff, Ark.

Put down any number of pounds, shillings and pence under £11, taking care that the number of pence is less than the number of pounds. Reverse this sum, putting pounds in the place of pence, and subtract from the original. Again reverse this remainder and add. The result in all cases will be £12 18s 11d, neither more nor less, whatever the amount with which we start. Verify and explain.

I. Solution by Prof. N. F. DAVIS, Brown University, Providence, R. I.; J. D. CRAIG, New Germantown, N. J.; the late SYLVESTER ROBBINS, North Branch Depot, N. J.; J. SCHEFFER, A. M., Hagerstown, Md.; COOPER D. SCHMITT. A. M., University of Tennessee, Knoxville, Tenn.; and H. C. WHITAKER, Ph. D., Manual Training School, Philadelphia, Pa.

	£	s.	d.
1st	a	b	c
Or thus	$a-1$	$19+b$	$12+c$
Reverse	c	b	a
Subtract	$a-c-1$	19	$12+c-a$
Reverse	$12+c-a$	19	$a-c-1$
Add	12	18	11

II. Solution by J. W. YOUNG, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y.; WALTER H. DRANE. Graduate Student, Harvard University, Cambridge, Mass.; and G. B. M. ZERR, A.M., Ph.D., The Temple College, Philadelphia, Pa.

The restrictions placed on the problem are unnecessary. The proposition is true for any sum of pounds, shillings and pence, provided, when we subtract we take care to take the positive difference of the two sums obtained as described. The proposition may be proved thus:

Let a, b, c be a number of pounds, shillings and pence, respectively, and suppose $a>c$.

	£	s.	d.	
The two sums are	a	b	c	....(1).
	c	b	a	....(2).

Since $a>c$, before subtracting we must write (1) in the form

$$a-1 \qquad 19+b \qquad 12+c$$

Subtract (2), and we have

	$a-c-1$	19	$12+c-a$
Reverse	$12+c-a$	19	$a-c-1$
Add	£12	18s.	11$d.$, the required result.

III. Solution by BENJAMIN F. YANNEY, A. M., Mount Union College, Alliance. O., and W. W. LANDIS, A. M., Dickinson College, Carlisle, Pa.

Let x stand for the number of pounds, y the number of shillings, and z the number of pence. Also, let $x>z$ and $<z+12$, Of course, $y<20$, and $z<12$.

Then performing the operations as indicated, we have,

1.	£x	ys.	zd.
	£z	ys.	xd.
	£$(x-1-z)$	19s.	$(z+12-x)$d.
2.	£$(x-1-z)$	19s.	$(z+12-x)$d.
	£$(z+12-x)$	19s.	$(x-1-z)$d.
	£12	18s.	11d.

NOTE.—In the case of the decimal system of notation, under similar conditions we find the result to be always 1089. In general, in any system of notation in which n units of the first order make one of the second order, and m units of the second order make one of the third order; if, also, x stands for the number in the third order, y the second order, and z the first order, where $x>z$ and $<z+n$, then proceeding as indicated in the problem, we shall always find the result to be n units of the third order, $n-2$ units of the second order, and $n-1$ units of the first order.

IV. **Solution by E. L. SHERWOOD, A. M., Professor of Mathematics, Beaver College, Beaver, Pa.**

This problem is an example of a class wherein the given quantities are caused to disappear by subtraction or cancellation—the resulting quantity depending on special circumstances, in this case the table ratios.

That the result will always be 12£ 18s 11d may be shown as follows:

	£	s.	d.
Given sum	a	b	c
Reversed	c	b	a
Remainder	$(a-1)-c$	19	$12+c-a$
Remainder reversed	$12+e-a$	19	$a-c-1$
Result	12	18	11

The sum need not be "under 11£" as was given in the problem, but up to a maximum of 23£ 19s 11d. More cannot be taken, as in the first subtraction we would have to "borrow" 2s instead of 1, or, in other words, use 24 as a ratio instead of 12.

If, however, we rigidly adhere to 20 and 12 as ratios, and use them, sums could be taken at random. As

£	s.	d.
100	30	25
25	30	100
74	19	−63
−63	19	74
12	18	11

A similar problem could be proposed with any table—the result varying with the constants of the table.

In general—if r and r' be the ratios (as 12 and 20) the result will be r of the highest, $r'-2$ of the next, and $r-1$ of the lowest denomination.

GEOMETRY.

135. Proposed by **WILLIAM HOOVER, A.M., Ph.D.,** Professor of Mathematics and Astronomy, Ohio University, Athens, O.

If a hyperbola be described touching the four sides of a quadrilateral which is inscribed in a circle, and one focus lie on the circle, the other focus will also lie on the circle.

Solution by the PROPOSER.

Using quadrilinear notation, the equation to the circle circumscribing the quadrilateral whose sides are given by $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, is $\alpha\gamma=\beta\delta$....(1).

Now, it is well known that if the coördinates of one focus of a conic tangent to a given line be α', β', γ', δ', those of the other focus are proportional to $1/\alpha'$, $1/\beta'$, $1/\gamma'$, $1/\delta'$.

But by the problem, α', β', γ', δ' is on (1); then $\alpha'\gamma'=\beta'\delta'$....(2), or

$$\frac{1}{\alpha'\gamma'}=\frac{1}{\beta'\delta'}\text{....(3).}$$

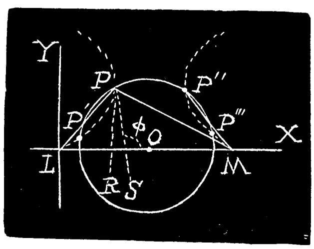
Substituting the reciprocals in (1) gives (3) also, and proves the theorem.

136. Proposed by **J. OWEN MAHONEY, B. E., M. Sc.,** Professor of Mathematics, Central High School, Dallas, Tex.

Construct a triangle having given the base, the median line to the base, and the difference of the base angles.

I. Solution by **B. L. REMICK,** Instructor of Mathematics, Bradley Institute, Peoria, Ill.

Let $LM=a=$the base, $CP=m=$median to the base, $\alpha-\beta=$difference of base angles.



Then vertex P of required triangle lies on circle about C (mid point of LM) as center with radius m; it also lies on the locus of point of intersection of straight lines through L, M forming angles with base having the required constant difference. We propose to show that this latter locus is an equilateral hyperbola and that our problem has therefore four solutions corresponding to the four common points of the circle and hyperbola.

$\angle RPS$ between the perpendicular and angle bisector$=\frac{1}{2}(\alpha-\beta)$ by a well known result in geometry; and hence we have to consider the locus of intersection of two straight lines passing through two given points L, M so that the angle bisector remains parallel to itself.

Let coördinates of P be (x_1, y_1).

Equation PL is $y_1x-x_1y=0$.

Equation PM is $y_1x+(a-x_1)y-ay_1=0$.

Equation PS is $\dfrac{y_1x-x_1y}{\sqrt{(y_1^2+x_1^2)}}=\dfrac{y_1x+(a-x_1)y-ay_1}{\sqrt{[y_1^2+(a-x_1)^2}}$

Since $\tan\phi$ has a constant value we obtain

$$\frac{\dfrac{y_1}{\sqrt{[y_1^2+(a-x_1)^2]}}-\dfrac{y_1}{\sqrt{(y_1^2+x_1^2)}}}{\dfrac{x_1}{\sqrt{(y_1^2+x_1^2)}}+\dfrac{a-x_1}{\sqrt{[y_1^2+(a-x_1)^2]}}}=k \text{ (constant).}$$

Simplifying, we have

$$(kx_1+y_1-ka)\sqrt{(y_1^2+x_1^2)}=(kx_1+y_1)\sqrt{[y_1^2+(a-x_1)^2]}.$$

Squaring and reducing, we obtain the form

$$2kx_1^2+2(1-k^2)x_1y_1-ky_1^2-2akx_1+(ak^2-a)y_1=0,$$

which shows the locus a conic.

Note points L, M, viz., $(0, 0)$, $(a, 0)$ are on locus.

Testing the discriminant to determine the nature of the conic, we have

$$AB-H^2=-4k^2-(1-k^2)^2=-(1+k^2)^2<0$$

Hence the curve is a hyperbola.

To examine the locus farther by proceeding to standard form of equation, make use of transformtion $x=x'+x_0$, $y=y'+y_0$, equate to zero coefficients of x', y' in the equation found and obtain as first reduced form

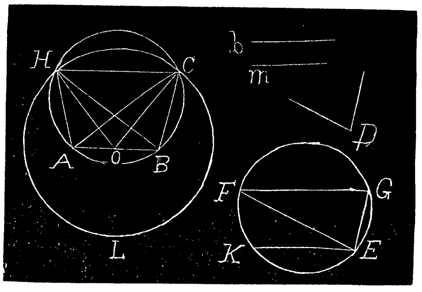
$$4kx'^2+4(1-k^2)x'y'-4ky'^2-a^2k=0.$$

Again using formulæ $A'+B'=0$, $A'B'=-(1+k^2)^2$ obtain as final reduced equation

$$(1+k^2)x''^2-(1+k^2)y''^2=a^2k,$$

indicating an equilateral hyperbola.

Solved in a similar manner by *H. C. WHITAKER.*



II. Solution by G. B. M. ZERR, A.M., Ph.D., The Temple College, Philadelphia, Pa.; ALOIS K. KOVARIK, Decorah Institute, Decorah, Ia.; and J. SCHEFFER, A. M., Hagerstown, Md.

Let b=base, m=median, D=difference of the base angles.

In any circle $FGEG$, find a segment $FKEG$ containing an angle FEG equal to $\angle D$ and draw EK parallel to FG.

Take $AB=b$; with O the center of AB as center and a radius equal to m, describe the circle HCL. Construct the angles ABH and HBC equal to the angles KEF and FEG, respectively.

Through $ABCH$ describe a circle. Join AH, OH, OC, AC. Then ABC or ABH is the required triangle, since $AB=b$, $OH=OC=m$, $\angle HCB=\angle HAC=D$.

CALCULUS.

102. Proposed by G. B. M.ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A right cone has its vertex at the focus of a paraboloid of revolution, the axis of the cone perpendicular to the axis of the paraboloid. Find the volume common to both.

Solution by the PROPOSER.

Let $x^2+y^2=c^2z^2$, be the equation to the cone. $y^2+z^2=4a(a+x)$, be the equation to the paraboloid.

The limits of z are $z=\dfrac{\sqrt{(x^2+y^2)}}{c}$ to $z=\sqrt{(4a^2+4ax-y^2)}$;

of y, 0 and $\sqrt{\dfrac{4a^2c^2+4ac^2x-x^2}{1+c^2}}=y_1$;

of x, $2ac[c-\sqrt{(1+c^2)}]=x_2$ to $2ac[c+\sqrt{(1+c^2)}]=x_1$.

$$\therefore\ V=2\int_{x_2}^{x_1}\left(\sqrt{(4a^2+4ax-y^2)}-\frac{\sqrt{(x^2+y^2)}}{c}\right)dxdy$$

$$=\int_{x_2}^{x_1}\Bigg[\bigg[y\sqrt{[4a^2+4ax-y^2]}+4a(a+x)\sin^{-1}\left(\frac{y}{2\sqrt{[a(a+x)]}}\right)-\frac{y\sqrt{[x^2+y^2]}}{c}$$

$$-\frac{x^2}{c}\log[y+\sqrt{(x^2+y^2)}]\bigg]_0^{y_1}\Bigg]dx=4a\int_{x_2}^{x_1}(a+x)\sin^{-1}\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}}{2\sqrt{[a(1+c^2)(a+x)]}}\right)dx$$

$$-\frac{1}{c}\int_{x_2}^{x_1}x^2\log\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}+c(2a+x)}{x\sqrt{(1+c^2)}}\right)dx.$$

$$\therefore\ V=\Bigg[2a(a+x)^2\sin^{-1}\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}}{2\sqrt{[a(1+c^2)(a+x)]}}\right)$$

$$-\frac{x^3}{3c}\log\left(\frac{\sqrt{[4ac^2(a+x)-x^2]}+c(2a+x)}{x\sqrt{(1+c^2)}}\right)\Bigg]_{x_2}^{x_1}+a\int_{x_2}^{x_1}\frac{x(a+x)dx}{\sqrt{[4ac^2(a+x)-x^2]}}$$

$$+\tfrac{1}{3}\int_{x_2}^{x_1}\frac{x^3dx}{\sqrt{[4ac^2(a+x)-x^2]}}-\frac{1}{3c}\int_{x_2}^{x_1}x^2dx$$

$$-\frac{1+c^2}{3e}\int_{x_2}^{x_1}\frac{x^4\,dx}{\sqrt{[4ac^2(a+x)-x^2]}\{c(2a+x)+\sqrt{[4ac^2(a+x)-x^2]}\}}.$$

Let $2ac^2-x=2ac\sqrt{(1+c^2)}\cos\theta$.

$$\therefore\ V=2a^3c\int_0^{\pi}[c-\sqrt{(1+c^2)}\cos\theta][1+2c^2-2c\sqrt{(1+c^2)}\cos\theta]d\theta$$

$$+\tfrac{8}{3}a^3c^2\int_0^{\pi}[c-\sqrt{(1+c^2)}\cos\theta]^3d\theta-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^{\pi}[c-\sqrt{(1+c^2)}\cos\theta]^2\sin\theta d\theta$$

$$-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^{\pi}\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}.$$

$$\therefore\ V=\frac{2\pi a^3c^2}{3}(6+15c^2+10c^4)-\tfrac{16}{9}a^3c^2\sqrt{(1+c^2)}(1+4c^2)$$

$$-\tfrac{8}{3}a^3c^2\sqrt{(1+c^2)}\int_0^{\pi}\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}.$$

Let $c=\cot\beta$.

$$\therefore\int_0^{\pi}\frac{[c-\sqrt{(1+c^2)}\cos\theta]^4d\theta}{\sin\theta-c\cos\theta+\sqrt{(1+c^2)}}=\frac{1}{\sin^3\beta}\int_0^{\pi}\frac{(\cos\beta-\cos\theta)^4d\theta}{1-\cos(\theta+\beta)}$$

$$=\frac{8}{\sin^3\beta}\int_0^{\pi}\sin^2\tfrac{1}{2}(\theta+\beta)\sin^4\tfrac{1}{2}(\theta-\beta)d\theta=D.$$

Let $\theta-\beta=2\phi$. $\therefore\ \theta+\beta=2\phi+5\beta$.

$$\therefore\ D=\frac{16}{\sin^3\beta}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}(\pi-\beta)}\sin^2(\phi+\beta)\sin^4\phi d\phi$$

$$=\frac{16}{\sin^3\beta}\int_{-\frac{1}{2}\beta}^{\frac{1}{2}(\pi-\beta)}(\sin^6\phi\cos^2\beta+2\sin\beta\cos\beta\sin^5\phi\cos\phi+\sin^4\phi\cos^2\phi\sin^2\beta)d\phi$$

$$=\tfrac{1}{2}\pi\operatorname{cosec}\beta(1+5\cot^2\beta)-\tfrac{2}{3}(1+4\cot^2\beta)=\tfrac{1}{2}\pi\sqrt{(1+c^2)}(1+5c^2)-\tfrac{2}{3}(1+4c^2).$$

$$\therefore\ V=\tfrac{2}{3}\pi a^3c^2(4+3c^2).$$

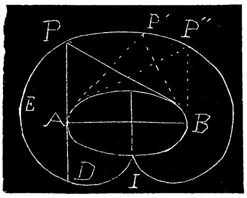
103. Proposed by HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A park, in the shape of an ellipse whose diameters are 100 and 50 rods, respectively, is surrounded by a wall: one end of a rope, whose length is the circumference of the ellipse, is fastened (outside of the wall) at one end of the longer diameter and the other end at the other end of the same diameter. Over how much surface will a horse graze, which is fastened to a ring moving freely on the rope ?

Remarks by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

No correct solution of this problem has yet been received. The problem has been misinterpreted by several of our contributors, who have mistaken it as

a corollary of Graves' Theorem. This was the interpretation according to which Dr. Haskell made his solution and which was inadvertently published in the MONTHLY, Vol. V, No. 4, page 111.



The problem as proposed is quite different and is not easy to solve, perhaps impossible.

The area grazed over consists of three parts :

(1) The part $ABP''P$—the area of this part is not difficult to find as the arc $PP'P''$ is the arc of a confocal ellipse; (2) two times the part $PEDAP$,—this part is generated by point, P, constrained to move under the composition of circular and elliptic evolutary motion ; and (3) two times the part $ADIA$, which is between the evolute of the elliptic field, the field, and the radius vector AD.

The part $PEDAP$ is the difficult part of the area to compute.

If any of our contributors will furnish a correct and complete solution of this problem it will be published in the next issue of the MONTHLY.

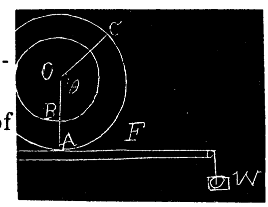
MECHANICS.

98. Proposed by WALTER H. DRANE. Graduate Student. Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, determine motion of spool and weight. [From problems in Mechanics at Harvard University.]

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let T=tension on thread, x=distance spool's center moves, y=distance weight W moves, θ=angle AOC through which spool turns, $OA=a$, OB the radius of the thread=b.



Then $x=a\theta$, $y=(a-b)\theta$.

The forces acting on the spool are friction, F; tension, T; and reaction, R, perpendicular to the table.

Let m=mass of spool and thread, m_1=mass of weight. Then

$$m\frac{d^2x}{dt^2}=T-F,\ mk^2\frac{d^2\theta}{dt^2}=Fa..(1,2),\text{ for spool};\ m_1\frac{d^2y}{dt^2}=T-m_1g..(3),\text{ for weight.}$$

Eliminating F between (1, 2) we get

$$ma\frac{d^2x}{dt^2}+mk^2\frac{d^2\theta}{dt^2}=T....(4)\ ;\ \text{but}\ \frac{d^2x}{dt^2}=a\frac{d^2\theta}{dt^2},\ \frac{d^2y}{dt^2}=(a-b)\frac{d^2\theta}{dt^2}.$$

ation, and weight will descend with an acceleration $a-b$ times the angular acceleration.

99. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

In a triangle ABC, base$=b$, area$=\triangle$, the principal moments of inertia at the centroid are $\frac{1}{72}m[a^2+b^2+c^2\pm\surd(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)]$ and the principal axes at this point make with the base AC an angle θ given by

$$\tan 2\theta=\frac{4(c^2-a^2)\triangle}{(a^2-c^2)^2-b^2(a^2+c^2)+2b^4}.$$

Solution by the PROPOSER.

Let O be the centroid and transform from the rectangular axes Ox, Oy, to the oblique axes Ox, OB.

Also let $(b/6AD)(2AD-3y)=x'$.

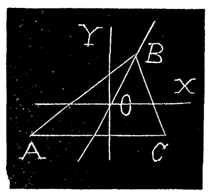
$$\text{Then } \Sigma mxy=\rho\sin^2 D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x'}^{x'}(x+y\cos D)y\,dy\,dx=\tfrac{1}{36}b\rho AD^3\sin^2 D\cos D$$

$$=\tfrac{1}{18}mAD^2\sin D\cos D.$$

$$\Sigma mx^2=\rho\sin D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x'}^{x'}(x+y\cos D)^2\,dy\,dx=\tfrac{1}{72}m(3b^2+4AD^2\cos^2 D).$$

$$\Sigma my^2=\rho\sin^3 D\int_{-\frac{1}{3}AD}^{\frac{2}{3}AD}\int_{-x'}^{x'}y^2\,dy\,dx=\tfrac{1}{18}mAD^2\sin^2 D.$$

$$\therefore\ \tan 2\theta=\frac{4AD^2\sin 2D}{3b^2+4AD^2\cos 2D}.$$



But $\sin D=a\sin C/AD$, $\cos D=\dfrac{4AD^2+b^2-4a^2}{4b.AD}$.

$$\sin 2D=\frac{a\sin C(4AD^2+b^2-4a^2}{2b.AD^2}=\frac{2(c^2-a^2)\triangle}{b^2.AD^2}.$$

$$\cos 2D=\frac{AD^2-2a^2\sin^2 C}{AD^2}=\frac{2(a^2-c^2)-2b^2(a^2+c^2)+b^4}{4b^2.AD^2}.$$

$$\therefore\ \tan 2\theta=\frac{4(c^2-a^2)\triangle}{(a^2-c^2)^2-b^2(a^2+c^2)+2b^4}.$$

$$A\cos^2\theta+B\sin^2\theta=\tfrac{1}{72}m(3b^2+4AD^2\cos^2 D).$$

$A\sin^2\theta+B\cos^2\theta=\frac{1}{18}mAD^2\sin^2D.$

$\therefore\ A+B=\frac{1}{72}m(3b^2+4AD^2).$

$(A-B)\cos2\theta=\frac{1}{72}m(3b^2+4AD^2\cos2D).$

$\therefore\ A=\frac{1}{72}m[a^2+b^2+c^2+2\sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}].$

$B=\frac{1}{72}m[a^2+b^2+c^2-2\sqrt{(a^4+b^4+c^4-a^2b^2-a^2c^2-b^2c^2)}].$

MISCELLANEOUS.

85. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that at least one of the three sides of a rational right triangle must be divisible by 5.

II. Solution by M. A. GRUBER, A.M., War Department, Washington, D. C.; HON. JOSIAH H. DRUMMOND, LL. D., Portland, Me.; and the PROPOSER.

Rational right triangles are divided into three kinds: (1), prime integral; (2), composite integral; and (3), fractional.

It is evident that prime integral right triangles are the basis of work.

To produce a prime integral right triangle we take *two integers prime to each other, one odd and the other even.* Then, twice their product will give one of the legs, the difference of their squares will give the other of the legs, and the sum of their squares will be the hypotenuse; or taking p and q as the two integers, the respective sides are $2pq$, p^2-q^2, and p^2+q^2.

This process will produce any prime integral right triangle.

The squares of even numbers end in 4, 6, and 0, and of odd numbers in 1, 9, and 5.

When the two prime integers, or p and q end, respectively, in 0 and an odd figure, or in 5 and an even figure, their product, or pq of the side $2pq$ contains the factor 5; the other sides being odd numbers *not* ending in 5.

When the squares of the two prime integers, or p^2 and q^2 end, respeetively, in 1 and 6 or in 9 and 4, the difference of these squares, or the side p^2 and q^2 ends in 5, and is, therefore, divisible by 5.

When the squares of the two prime integers end, respectively, in 1 and 4 or in 9 and 6, the sum of these squares, or the side p^2+q^2 ends in 5.

The above contains all the combinations of any two prime integers, one odd and the other even, according to the formation of the sides of prime integral right triangles.

$\therefore$ In every *prime* integral right triangle one of the sides and *only one* is divisible by 5.

In composite integral and fractional right triangles either *one* or *all* of the sides must be divisible by 5. For, in order to have more than one side divisible by 5, the highest common factor of the three sides must contain the factor 5.

$\therefore$ In rational right triangles either *one* or *all* of the sides will be divisible by 5.

86. Proposed by **GEORGE LILLEY**, Ph. D., LL. D., Professor of Mathematics, University of Oregon, Eugene, Oregon.

A gave two notes; one for a dollars at m per cent., and the other for b dollars at n per cent., annual interest. He is to make a monthly payment of c dollars. How much must be endorsed on each note in order to pay them off at the same time? What must be the payment on each if $a=1900$, $b=1800$, $m=6$, $n=7$, and $c=25$?

Solution by **WM. FRED FLEMING**, Principal of High School, Denison, Texas.

One general formula for finding the number of fixed monthly payments required to extinguish a debt which bears interest is as follows:

$$\text{Number of payments}=\frac{\log\left(\dfrac{p}{p-mP}\right)}{\log(m+1)}$$

where p=payment, P=principal, and m=fraction of principal accruing as monthly interest.

In the given problem the number of payments is equal since the notes are to be lifted at the same time. Hence

$$\frac{\log\left(\dfrac{p}{p-10.50}\right)}{\log(\frac{7}{1200}+1)}=\frac{\log\left(\dfrac{25-p}{25-p-9.50}\right)}{\log(\frac{1}{200}+1)}.$$

Whence, using 7-figure logarithms,

$$\frac{p}{p-\frac{21}{2}}=\left(\frac{25-p}{\frac{31}{2}-p}\right)^{1.1662}$$

which equation, solved by double position (or some other method of approximation) gives, as the nearest payments involving *even cents*, \$12.74 for the payment on the \$1800 note, and \$12.26 as payment on the \$1900 note.

Number of payments=298.9, or 24 years and 11 months (nearly) will be required to lift the note.

PROBLEMS FOR SOLUTION.

ARITHMETIC.

136. Proposed by **F. M. PRIEST**, Mona House, St. Louis, Mo.

What is the size of the smallest cubical box, inside dimension, that will contain four balls each ten inches in diameter?

137. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

At the corners of a triangle sides a, b, c feet are towers d, e, f feet high. At what point must a ladder be placed so that it will just reach to the top of each tower without moving? How long is the ladder? Substitute $a=200$, $b=180$, $c=150$, $d=60$, $e=50$, $f=30$; d, e, f at A, B, C, respectively.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

ALGEBRA.

125. Proposed by LESLIE L. LOCKE, Instructor in Mathematics, Michigan Agricultural College, Ingram County, Mich.

What special expedient will solve the system

$$\left.\begin{array}{l} x^4-y^4=369 \\ x-y=1 \end{array}\right\} ?$$

126. Proposed by CHARLES C. CROSS, Meredithville, Va.

A and B run a race; B, who runs slower than A by a miles in b hours, starts first by c minutes, and they get to the n-mile stone together; required their rates of running. If $a=1$, $b=2$, $c=2$, and $n=4$, what is the result?

*** Solutions of these problems should be sent to J. M. Colaw not later than Jan. 10.

GEOMETRY.

154. Proposed by ALFRED HUME, C. E., D. Sc., Professor of Mathematics, University of Mississippi, P. O., University, Miss.

The angle between the edge of a trihedral angle and the bisector of the opposite face angle is less than, equal to, or greater than, half the sum of the other two face angles, according as it is itself acute, right, or obtuse.

155. Proposed by J. C. NAGLE, M. A., M. C. E., Professor of Civil Engineering, State Agricultural and Mechanical College, College Station, Tex.

A special case of the following problem was sent me some time ago by an ex-member of one of my engineering classes, as occuring on the Southern Pacific Ry. near Devil's River:

Two straight tracks, p feet between centers, are to be united by a cross-over composed of two curves of radius R, and a length L of intervening tangent. Required the central angles and the distance between tangent points, measured along main track. In the special case referred to p was 62 feet, L 100 feet with $9^\circ\ 30'$ curves.

156. Proposed by F. M. McGAW, A. M., Professor of Mathemstics, Bordentown Military Institute, Bordentown, N. J.

To construct an equilateral triangle such that its vertices shall be in each of two parallel lines and a point fixed between these lines.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

CALCULUS.

116. Proposed by JOHN M. COLAW, A. M., Monterey. Va.

"Prove that the length of the *greatest* beam of square section that can be cut from a log l feet long and in the shape of a conic frustum, diameters D and d, is $\frac{1}{3}lD\div(D-d)$ feet."

117. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A frustum of a paraboloid of revolution closed at both ends has a given volume. Find its interior dimensions when its surface is a minimum.

118. Proposed by J. W. YOUNG, Oliver Graduate Student, Cornell University, Ithaca, N. Y.

Find the differential equations of the system of parabolas, $y^2=4a^2(x+a^2)$, and of its orthogonal trajectories, and interpret the result. Find also the equation of the system of trajectories.

*** Solutions of these problems should be sen to J. M. Colaw not later than Jan. 10.

MECHANICS.

105. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A man, weighing 150 pounds at the surface of the earth, ascends in a balloon, until the area visible to him is $2\pi R^2(1-\frac{1}{2}\sqrt{2})$. What is his weight at that height ?

106. Proposed by F. P. MATZ. M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg. Pa.

Vary the radius of curvature of a plane curve inversely as the abscissa; then the solution will give you, (1) Ryan's Equation of the Elastic Curve, and (2) Wood's Equation of the Hydrostatic Curve.

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

AVERAGE AND PROBABILITY.

97. Proposed by L. C. WALKER, A. M., Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

A straight line is drawn at random across a circle, and five points are taken at random in the surface of the circle. Required the chance that all the points are on the same side of the line.

98. Proposed by REV. PREBENDARY WHITWORTH, A. M.

A has £m and B has £n. They play for points until one of them has lost all his money. If α and β be the respective chances that A and B win any point, the expectation of the number of points played will be

$$\frac{n\alpha^n(\alpha^m-\beta^m)-m\beta^m(\alpha^n+\beta^n)}{(\alpha-\beta)(\alpha^{m+n}-\beta^{m+n})}.$$

*** Solutions of these problems should be sent to B. F. Finkel not later than Jan. 10.

MISCELLANEOUS.

98. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A particle describes an ellipse under an attraction always directed to the vertex; to determine the law of the attraction.

99. Proposed by F. P. MATZ, M. Sc., Ph. D., Professor of Mathematics and Astronomy, Irving College, Mechanicsburg, Pa.

Through the zenith of an observer at the sea-coast in north latitude $\varphi=40^\circ$, a "cat's-tail cloud," height $h=10$ miles, extends northeast until it touches the horizon. How far from the observer is the *advance-end* of the cloud? What is the length of the cloud measured from the end specified to the observer's zenith?

⁂ Solutions of the problems should be sent to J. M. Colaw not later than Jan. 10.

EDITORIALS.

Prof. B. O. Peirce, of Harvard University, has been granted a year's leave of absence.

Dr. G. A. Bliss has been appointed Instructor in Mathematics at the University of Minnesota.

Prof. Irvin Stringham has returned to the University of California after a year of travel in Europe.

Prof. W. Dyck has been appointed Director of the Technical High School in Munich for the next three years.

Prof. R. E. Allardice, of Leland Stanford Jr. University, is spending a year abroad on leave of absence.

Prof. E. J. Townsend has returned to the University of Illinois, after two years of study at the University of Göttingen.

Owing to our engravers having sent the plate of Mr. Tucker's portrait to some other Springfield, and which has not yet been returned to us, we are obliged to have it appear in next issue.

BOOKS AND PERIODICALS.

Logarithmic and Trigonometric Tables, Five-Place and Four-Place. Edited by D. A. Murry, Cornell University. 8vo. Cloth, 96 pages. Price, 54 cents. New York: Longmans, Green & Co.

This book contains Five-Place Logarithms of Numbers; Five-Place Logarithms of the sine, cosine, tangent and cotangent for each minute from 0° to 90°; also Four-Place Logarithms corresponding to the Five-Place Logarithms of Number and the Trigonometric Functions. The editor believes that the proportional parts should be calculated, and not copied, by those who use logarithms and trigonometric tables for the first time. So these tables do not have the proportional parts set down in the tables. We heartily concur with him in this opinion. B. F. F.

Notes on the History of American Text-Books on Arithmetic. By J. M. Greenweed, LL. D., Superintendent of Schools, Kansas City, Mo., and Artemas Martin, A. M., Ph. D., LL. D., Washington, D. C. Issued by the National Bureau of Education.

The pamphlet before us is a chapter from the Report of the Commissioner of Education, and comprises pages 781—838 of the Report. The titles of over 250 text-books on Arithmetic are given, together with the names of the authors. The earliest published Arithmetic in the list here given is that of Reffelt's, published in 1861. Earlier Arithmetics are given in a previous chapter. In many instances short biographical sketches of the authors are given. This bibliography will, when complete, be of great value. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

The world has never known a more dramatic situation than that presented by the foreign community within the walls of Peking while cut off from communication with their countrymen. During these long, doubtful weeks, the most interesting figure in this international tragedy was Sir Robert Hart, who for more than twenty-five years has been as far, as a European might, the statesman guiding the affairs of the Chinese Empire. Those familiar in any degree with Eastern conditions hoped, after the relief of Peking, that Sir Robert would break his long rule of silence and give to the world his story of the events which led to the closing of the gates of the British Legation, and his views as to the policies which should prevail in the settlement of difficult questions which have arisen. On the 17th of October, the following cable-message from Sir Robert's London representative to the editor of *The Cosmopolitan* was received: "Sir Robert Hart has sent for November number *Fortnightly* London, and *Cosmopolitan*, New York, an important article on siege of Peking, about fifteen thousand words, which I will post you to-morrow."

The MS. arrived in time to be included in the December issue. It will be read with the deepest interest, both by statesmen and the general public. *The Cosmopolitan* has been highly honored by Sir Robert Hart in his selection of the American magazine through which this valuable contribution to the history of the world is given publicity. B. F. F.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine, edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., New York.

The November number contains interesting articles on the election. In addition, the following valuable articles are to be found in it: The Great Growth of Trusts in England, The Political Beginning in Porto Rico, and the Hall of Fame for Great Americans.
B. F. F.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

This magazine gives a brief resume of all the principal events of the month, which take place anywhere in the civilized world. It is just the journal for the busy man.
B. F. F.

INDEX.

QUERY. When $a=b$, find value of $\dfrac{ab[-b\pm\sqrt{2a^2-b^2}]}{a^2-b^2}$?

SOLUTIONS AND PROBLEMS.

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ALGEBRA.

GEOMETRY.

CALCULUS.

MECHANICS.

DIOPHANTINE ANALYSIS.

AVERAGE AND PROBABILITY.

MISCELLANEOUS.

THE
AMERICAN
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VOL. VII. DECEMBER, 1900. No. 12.

SOME METHODS OF CONSTRUCTING SUBSTITUTION GROUPS.

By DR. G. A. MILLER, Cornell University, Ithaca, N. Y.

We have observed that the first n powers of a circular substitution of degree n constitute a group of order n. If a substitution is composed of more than one cycle its order is the least common multiple of the degrees of all its cycles. This is also the order of the group which it generates. Any group which is generated by a single substitution is called *cyclical*. A cyclical group is clearly abelian but an abelian group is not necessarily cyclical. *There is one and only one cyclical abstract group of every possible order n* since the combinatory laws of such a group are the same as those of the roots of the equation $x^n=1$, where n is any positive integer. The only orders of which there is no group except the cyclical are those which are not divisible by the square of a prime and do not contain any prime factor which is congruent to 1 with respect to another prime factor as modulus. The composite orders below 100 that have this property are 15, 33, 35, 51, 65, 69, 77, 85, 87, 91, 95.

Let S and T represent any two commutative substitutions. This property is expressed by each of the following three equivalent equations:

$$ST=TS \qquad T^{-1}ST=S \qquad T=S^{-1}TS$$

where T^{-1} and S^{-1} indicate the inverse of T and S, respectively; *i. e.* $T^{-1}T=1$ $=S^{-1}S$. In general $T^{-1}ST$ is called the transform of S with respect to T. All the substitutions of degree m that are commutative with any given substitution T

of degree n form a group, for if S_1 and S_2 are commutative with T then will S_1^2, S_2^2, S_1S_2, S_2S_1 have the same property.* By raising both members of the equation $S^{-1}TS=T$ to the β power we have

$$S^{-1}TS.S^{-1}TS.S^{-1}TS\ldots.=T^\beta \text{ or } S^{-1}T^\beta S=T^\beta .$$

Hence every substitution that transforms T into itself must transform each of the substitutions of the group generated by T into itself. When T consists of a single cycle m cannot be less than n,† and when $m=n$ all the substitutions that are commutative with T are the n different powers of T; for if some other substitution of degree n would transform T into itself its product into some power of T would give a substitution of degree less than n that would also transform T into itself.

If T consists of β cycles of degree $n\div\beta$ (in this case T is said to be a regular substitution) the order of the group of degree n which is formed by all the substitutions that are commutative with T is $\left(\frac{n}{\beta}\right)^\beta.\beta\,!$ since the β cycles are permuted according to the symmetric group of degree β. In general, let T contain a_1 cycles each of degree a_2, b_1 cycles each of degree b_2, ...; then the order of the group which is formed by all the substitutions of degree n that are commutative with T is $a_2^{a_1}.a_1!.b_2^{b_1}.b_1!\ldots.$ In particular, when no two cycles of T are of the same degree the order of this group is the product of the degrees of these cycles, and the necessary and sufficient condition that the powers of T include all the substitutions of degree n that are commutative with T is that the degree of each cycle of T is prime to that of every other cycle. When $m=n+\alpha$ $(\alpha>1)$ the required group is obtained by multiplying the given group of degree n by the symmetric group of order $\alpha\,!$

We have now considered the groups which are formed by all the substitutions which are commutative with a given substitution T; *i. e.* by all the values of S which satisfy the equation $S^{-1}TS=T$. This is a special case of the problem to find all the groups which are formed by all the values of S which satisfy the equation $S^{-1}TS=T$.

By raising both members of this equation to the β power we obtain $S^{-1}T^\beta S=T^\beta=(T^\beta)^\alpha$; *i. e. if a substitution transforms T into a certain power it transforms all the substitutions of the group generated by T into the same power*, and hence it must transform this group into itself. Since all the substitutions which transform a group into itself must form a group all the values of S which satisfy the equation $S^{-1}TS=T^\alpha$, α having every possible value, constitute a group (G) whose order we proceed to determine.

From $(S^{-1}TS)^\alpha=S^{-1}T^\alpha S$ and $S^{-1}\mid S=1$ it follows that a transform of T cannot be of a higher order than T, and from $S^{-1}T^\alpha S=1$ we have $T^\alpha S=S$ or

*Cf. Burnside, ***Theory of Groups***, 1897, page 215.

†It will be assumed throughout that the substitutions of degree m which are commutative with T involve no elements except those contained in T whenever m is equal to or less than n. When m is greater than n these substitutions are supposed to include all the elements of T.

$T^{\alpha}=1$; *i. e.* a transform of T cannot be of a lower order than T. Since the order of any transform of T is equal to the order of T it follows that α must be prime to the order of T in $S^{-1}TS=T^{\alpha}$. By writing any such power of T below T we can at once write down the substitution which transforms T into this power. Hence α can have all the values that are prime to the order of T. To complete the determination of the order of G it is desirable to arrange its substitutions in a rectangular form, as follows :*

$$\begin{array}{llll}
1 & s_1 & s_2 \ldots\ldots\ldots s_{l-1} \\
t_1 & s_1t_1 & s_2t_1 \ldots\ldots s_{l-1}t_1 \\
t_2 & s_1t_2 & s_2t_2 \ldots\ldots s_{l-1}t_2 \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot \\
\cdot\quad\cdot & \cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot \\
t_{k-1} & s_1t_{k-1} & s_2t_{k-1} \ldots s_{l-1}t_{k-1}
\end{array}$$

The first row contains all the substitutions which transform T into itself and hence forms a subgroup of G. It is supposed that each t is so selected that it does not occur in any row that precedes it. Since the first row includes all the substitutions of G that transform T into itself $t_1^{-1}Tt_1=T^{\alpha_1}$ (α_1 is not equal to 1). Hence $(s_\alpha t_1)^{-1}Ts_\alpha t_1=t_1^{-1}s_\alpha^{-1}Ts_\alpha t_1$†$=t_1Tt_1=T^{\alpha_1}$ ($\alpha_1=0, 1, 2, \ldots, l-1$) ; *i. e.* each of the substitutions of the second row transforms T into T^{α_1}. G cannot contain any other substitution h which has this property for by transforming by t_1^{-1} each member of the equation $h^{-1}Th=T^{\alpha_1}$ we have $(ht_1^{-1})^{-1}Tht_1^{-1}=t_1T^{\alpha_1}t_1^{-1}=T$. From the fact that $ht_1^{-1}=s_\alpha$ it follows that $h=s_\alpha t_1$. No two of the substitutions of the second row are identical for from $s_\alpha t_1=s_\beta t_1$ there results $s_\alpha t_1t_1^{-1}=s_\beta t_1t_1^{-1}$ or $s_\alpha=s_\beta$. The number of substitutions that transform T into T^{α_1} is therefore equal to l, the number that are commutative with T. The order of G is then kl, k being equal to the number of positive integers less than the order of T and prime to it.

Having determined the order of the groups which are composed of all the substitutions that are commutative with a given substitution T or with the group generated by T, we proceed to determine some of the properties of these groups. We first observe that when T is regular the group which is composed of all the substitutions which transform T into itself contains no selfconjugate substitutions except the powers of T. Since such a group contains the product of all the cycles of T a selfconjugate subgroup could not permute any of the cycles of T, and among the substitutions that do not permute any of these cycles the powers of T are clearly the only ones that are selfconjugate. This fact leads to the following interesting result : *The necessary and sufficient condition that the group of degree n which is composed of all the substitutions which are commutative to T is*

*This arrangement seems to have been first employed by Abbati in 1802, *Burkhardt, Zeitschrift fur Mathematik und Physik*, Vol. 37, page 141. By means of it we can see directly that the order of any subgroup is a divisor of the order of the group.

†$(t_1t_2\ldots t_r)^{-1}=t_r^{-1}\ldots t_2^{-1}t_1^{-1}$ since $t_1t_2\ldots t_rt_r^{-1}\ldots t_2^{-1}t_1^{-1}=1$.

conjugate to that which is composed of all the substitutions that are commutative to T' is that T and T' are conjugate.*

When an element of a substitution is replaced by itself it is said to form a cycle of a single element. *E. g.* the substitution *abc.de* is said to consist of two cycles if it is regarded as a substitution of degree five. If it is regarded as a substitution of degree six it is said to consist of three cycles, etc. According to this definition of cycle the product of any substitution s and a transposition† t involves either one more cycle or one less cycle than s. First suppose that the elements of t are found in two cycles of s and that each of these cycles begins with an element of t. It is clear that st contains a single cycle which includes all the elements of these two cycles and hence st has one less cycle than s. The other possibility is that the two elements of s are found in the same cycle of s and we may suppose that this cycle c begins with one of these elements. Let $c=a_1 \ldots . a_\alpha \ldots . a_\beta$ and $t=a_1 a_\alpha$ then will $ct=a_1 \ldots . a_{\alpha-1}.a_\alpha \ldots . a_\beta$, the subscripts of the elements of c being arranged in ascending order and $\alpha \overline{\gtrless} 2$, $\beta \overline{\gtrless} \alpha$. Hence we observe that in this case st contains one more cycle than s.

Any cycle and hence any substitution can be represented as the product of a series of transpositions in an infinite number of ways. For instance, $c=a_1 a_2 a_3 \ldots . a_l=a_1 a_2 \times a_1 a_3 \times \ldots . \times a_1 a_l=a_1 a_3 \times a_3 a_2 \times a_1 a_4 \times \ldots . a_1 a_l=$etc.

By means of the results in the preceding paragraph we can readily prove that the numbers of these transpositions, for a given substitution s, are either all odd or all even. For, let $s=t_1 t_2 \ldots . t_r$, t_1, t_2, ..., t_r being transpositions. Then will $st_r \ldots . t_2 t_1=1$. If γ is the number of cycles of s and n its degree the first member of the last equation involves $n-\gamma$ more cycles than s. Hence $r-(n-\gamma)$ must be even; *i. e.* if one of the two numbers r, $n-\gamma$ is even the other is even also. Since the latter of these numbers is fixed when the substitution is given the theorem‡ is proved. Incidentally we observe that $n-\gamma$ is the smallest number of transposittons whose product is a substitution of degree n and γ cycles and that $r-(n-\gamma)$ can be made equal to any even integer. s is called positive or negative as r is even or odd.

Since the product of two positive substitutions is positive it follows that every group which involves negative substitutions must contain a subgroup composed of its positive substitutions. The order of this subgroup is half the order of the entire group since the product of a positive and a negative substitution is negative and the product of two negative substitutions is positive. The group which is composed of all the positive substitutions of degree n is called the alternating group of order $n! \div 2$. If a subgroup is transposed into itself by all the substitutions of the entire group it is called *selfconjugate*. A subgroup which includes half the substitutions of a group must be either selfconjugate or one of two conjugate subgroups. The latter is impossible since all the substitutions of one

*Two substitutions, or groups, are conjugate when one is the transform of the other.

†A transposition is a substitution of degree two; *e. g.*, *ab*, *cd*, ...

‡For a different proof of this fundamental theorem see Burnside, ***Theory of Groups***, 1897, page 9; Cole's edition of ***Netto's Theory of Substitution***, 1892, page 17; etc.

of these conjugates would have to transform the other into itself for a substitution must transform every subgroup in which it occurs into itself.

While every subgroup whose order is half the order of the group is self-conjugate a subgroup of any lower order need not be self-conjugate. If a group contains a subgroup of one-third its order that is not selfconjugate it must contain three subgroups of this order which are transformed according to the symmetric group of order six by all the substitutions of the group.

AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS. Ph. D., Stevens Point, Wis.

[Concluded from November Number.]

APPLICATION TO MECHANICS.

168. A force is completely represented by a point vector. We will denote a force by $F\rho$, where F denotes the length and direction of a vector, indicating respectively the intensity and direction of the force, and ρ is the point of application. It is apparent here that two letters are needed to properly represent the complex concept of a force. (See 76).

158. In Chapter I, we saw that the sum of two vectors is the diagonal of a parallelogram whose adjacent sides are the two given vectors. Then the sum of two forces, or their resultant, is the diagonal of a parallelogram whose two adjacent sides represent the two given forces. Similarly, all the results obtained for vectors in that chapter hold equally well for forces. The condition for equilibrium of forces acting on a particle at the extremity of ρ is evidently $(\Sigma F)\rho=0$, or $\Sigma F=0$.

169. The formulas obtained in Chapter VI evidently hold true when the points are replaced by infinitesimal forces, as parallel forces acting on particles, and also when they are replaced by finite parallel forces. (See 80).

170. Let the resultant of the forces acting on a rigid body be denoted by R. Then if $\varepsilon=\rho-\rho_1$

$$R=\Sigma F\rho=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=(\Sigma F)\rho_1+(\Sigma F\varepsilon).$$

This result, called a "Wrench," contains two parts, a vector and a plane segment part. The vector ΣF represents the translation force, and the plane segment $\Sigma F\varepsilon$ gives the plane and magnitude of rotation. When the above result is interpreted geometrically, *i. e.* when ΣF is thought of as a line, and $\Sigma F\varepsilon$ as a plane segment, R is called a "Screw."

171. If R reduce to a single force, then by 34, $R^2=0$. We have then

$$R^2=(\Sigma F\rho_1+\Sigma F\varepsilon)^2=2(\Sigma F.\Sigma F\varepsilon)\rho_1=0.$$

This shows that ΣF and $\Sigma F\varepsilon$ must be parallel (112), *i. e.* the resultant force is parallel to the plane of the resultant couple. Evidently $\Sigma F.\Sigma F\varepsilon=0$ is satisfied also either by $\Sigma F=0$ whence $R=$a couple, or by $\Sigma F\varepsilon=0$, which makes of R a single force. If all of the forces lie in a single plane, the resultant can be reduced to either a single force or to a couple.

172. *For equilibrium we must have* (170), $\Sigma F=0$ *and* $\Sigma F\varepsilon=0$.

173. *A total resultant effect may be reduced to a single force and a couple whose place is perpendieular to the force by properly choosing the point of application.*

PROOF.

$$R=\Sigma F\rho_1+\Sigma F(\rho-\rho_1)=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+\Sigma F(\rho-\rho_1).$$

Let $\Sigma F(\rho-\rho_1)=|\varepsilon$. Then

$$R=\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1)+|\varepsilon.$$

Now the condition that $\Sigma F\rho_2$ shall be perpendicular to $|\varepsilon-\Sigma F(\rho_2-\rho_1)$ is

$$(|\varepsilon-\Sigma F(\rho_2-\rho_1)|\Sigma F\rho_2=|\varepsilon|\Sigma F\rho_2-\Sigma F(\rho_2-\rho_1).|\Sigma F\rho_2=0.$$

$\therefore |\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F=-|\Sigma F\rho_2\varepsilon$, (64, 38), whence

$$\frac{|\Sigma F\rho_2.\Sigma(\rho_2-\rho_1)F}{(\Sigma F\rho_2)^2}=-\frac{|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}.$$

Comparing the left member of this equation with the formula of 132, we see that the right member is the value of the orthogonal projection of $\rho_2-\rho_1$ on a plane perpendicular to $\Sigma F\rho_2$.

Multiplying $\Sigma F\rho_2$ by the members of the last equation and interchanging the factors of $\Sigma(\rho_2-\rho_1)F$, we get

$$\Sigma F(\rho_2-\rho_1)=\frac{\Sigma F\rho_2|\Sigma F\rho_2\varepsilon}{(\Sigma F\rho_2)^2}=\frac{|\Sigma F\rho_2.\Sigma F\rho_2|\varepsilon}{(\Sigma F\rho_2)^2}-|\varepsilon \text{ (144, last equation).}$$

Substituting the last value of $\Sigma F(\rho_2-\rho_1)$ in the first equation of this article, we have

$$R=\Sigma F\rho_2+\frac{\Sigma F\rho_2\Sigma F(\rho_2-\rho_1)}{(\Sigma F\rho_2)^2}.|\Sigma F\rho_2,$$

which gives the required reduction.

CHAPTER XIV.

APPLICATION TO LOGIC.

174. The law of the inner product is $e_r e_s = 0$. For in 117 if F is different from E, $[E|F]$ contains equal factors and is therefore zero, (43). This law, which is the opposite of that of the outer product (34), is made the basis of the study of spaces. Now a space in the *Ausdehnungslehre* corresponds to a *concept* or notion in logic. Hence the former science can be applied in the latter.

175. Definition.—The *Combined Space*, or the *sum* of two spaces, is the totality of quantities which belong to one or other of them (17).

176. Definition.—The *Common Space*, or the *product* of two spaces, is the totality of quantities which are common to both, (see 104). The product of two spaces which have no quantity of the first order common is zero (174).

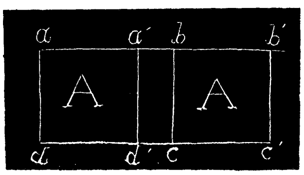
Thus, using $°(L_1 L_2)$ and $°(L_1 L_3)$ to denote the two spaces which contain these lines, we see that L_1 is the common space.

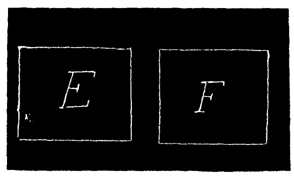
177. *The sum of the orders m and n of two spaces equals the sum of the orders p and q of their common and combined spaces.*

Evidently $m+n$ duplicates the number expressing the order of the common space, and $p+q$ does the same.

178. *All the laws of space analysis continue to hold true in logic when the word "space" is replaced by the logical term "concept."*

Proof.—It is evident from the definitions 175, 176 that $°(e+e) = °e$, and $°(e.e) = °e$, and also that $°e_r + °e_s$ is greater or less than 0 and $°(e_r).°(e_s) = 0$, when r is greater or less than s. But these are likewise the basic formulas of logic. (See H. and R. Grassmanns' *Formenlehre*, B. II., *Die Begriffslehre, oder Logik*,





1872, page 43. See also Encyclopedia Britannica article 'Logic,' section 35, paragragh 6.)

179. Two spaces can contain equal simple quantities only when they overlap. Thus if $A = abcd$ and $A' = a'b'c'd'$, these spaces have $a'bcd'$ in common, and quantities in $a'bcd'$ are in both A and A'. But E and F have no simple quantities in common. Since they are of the same size and lie in the same plane E and F in *geometry* would be equal (94); but in the theory of space or logic they are altogether different, having not one point, element, or bit of surface in common. It is highly important to note this difference if the reader is to avoid misconception.

180. *The associative and commutative laws for addition and multiplication hold for space analysis.*

PROOF.—It is evident from the definition 175 that $a+b=b+a$. Similarly from the definition in 176 it follows that $ab=ba$, and $abc=a(bc)$.

181. *Every sum, $n.^\circ A$, or product, $(^\circ A)^n$, formed from the same space, $^\circ A$, equals this space.* This follows from proof in 178.

182. *One can add to any space a product of two spaces, one of whose factors is the given space, without altering its value.* Thus

$$^\circ A = {}^\circ A + {}^\circ A\, ^\circ B.$$

One can multiply any space by a sum of two spaces, one of whose parts is the given space, without altering its value. Thus,

$$^\circ A = {}^\circ A(^\circ A + {}^\circ B).$$

PROOF.—The product $^\circ A\, ^\circ B$ is that part of $^\circ A$ which is common to both factors, 176. But adding a part of $^\circ A$ to $^\circ A$ gives $^\circ A$ by 181. In the other case we see that what is common to $^\circ A$ and $^\circ A + {}^\circ B$ is $^\circ A$.

183. *If $^\circ A + {}^\circ B = {}^\circ B$, $^\circ A . {}^\circ B = {}^\circ A$.*

PROOF.—$^\circ A . {}^\circ B = {}^\circ A(^\circ A + {}^\circ B)$ (Hyp.)
$= {}^\circ A$ (82)

If $^\circ A . {}^\circ B = {}^\circ A$, $^\circ A + {}^\circ B = {}^\circ B$.

PROOF.—$^\circ A + {}^\circ B = {}^\circ A\, ^\circ B + B^\circ$ (Hyp.)
$= {}^\circ A\, ^\circ B + {}^\circ B^2$ (181)
$= {}^\circ B(^\circ A + {}^\circ B$ (Dist. Law)
$= {}^\circ B$ (182)

184. *Unity added to any space gives unity, and any space multiplied by unity gives the same space; nought added to any space gives the same space, and any space multiplied by nought gives nought.*

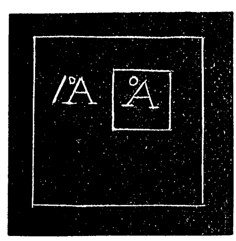
PROOF.—The last three of these results are self-evident. From the first we have

$$1 = 1(1 + 1 \times {}^\circ A) \qquad (182)$$
$$= 1 + {}^\circ A \qquad \text{(Dist. law and 2. of this Article.)}$$

185. *If $^\circ A + {}^\circ C = {}^\circ B + {}^\circ C$ and also $^\circ A . {}^\circ C = {}^\circ B . {}^\circ C$, $^\circ A = {}^\circ B$.*

PROOF.—$^\circ A = {}^\circ A(^\circ A + {}^\circ C)$ (182) $= {}^\circ A(^\circ B + {}^\circ C)$ (hyp.) $= {}^\circ A . {}^\circ B + {}^\circ A\, ^\circ C$ (Dist. law)
$= {}^\circ A\, ^\circ B + {}^\circ B\, ^\circ C$(hyp.) $= {}'B(^\circ A + {}^\circ C) = {}^\circ B(^\circ B + {}^\circ C)$(hyp.) $= {}^\circ B$ (182).

DEFINITION.—The Non-Space or Complementary Space of $^\circ A$ is that space $|^\circ A$ (read non-A) which contains all the units not found in $^\circ A$, and none of the units which are found in $^\circ A$.



187. *The sum of a space $^\circ A$ and $|^\circ A$ is unity: the product of a space $^\circ A$ and $|A$ is zero.*

PROOF.—The truth of the second part of the theorem follows directly from 176. For the other we write, by 184,

$$^\circ A + |\,^\circ A = (^\circ A + |\,^\circ A)\,|.$$

But by 176 the product of two spaces cantains all the quantities which are common to both. Then all the quantities of the left member $^\circ A + |\,^\circ A$ are contained in $|$. But $^\circ A$ and $|\,^\circ A$ contain all the units there are. Then $|$ contains all possible units.

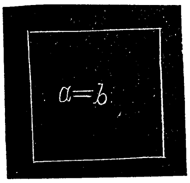
188. *All non-spaces of the same spaces are equal.*

Proof.—Suppose $|^{\circ}A$ and $|^{\circ}A_1$ to be two non-spaces of $^{\circ}A$. Then $^{\circ}A+|^{\circ}A=1=^{\circ}A+|^{\circ}A_1$; whence by 185, remembering that $^{\circ}A.|^{\circ}A=0$ and $^{\circ}A.|^{\circ}A_1=0$, $|^{\circ}A=|^{\circ}A_1$.

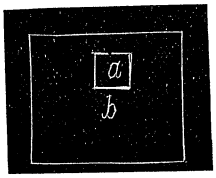
189. *The non-space of the non-space of a given space is the given space.*

Proof.—We have $|^{\circ}A+||^{\circ}A=1=|^{\circ}A+^{\circ}A$, and $||^{\circ}A.|^{\circ}A=0=^{\circ}A\,|^{\circ}A$ (187). Whence, by 185, $||^{\circ}A=^{\circ}A$.

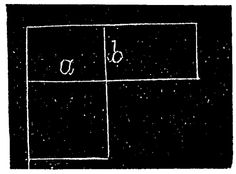
190. We give below four diagrams with descriptive names.



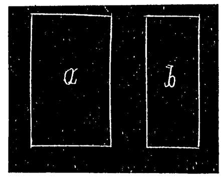
Identical Spaces.



Incident Space.



Cutting Spaces.



Disjunctive Spaces.

191. *Each of two spaces is incident to their sum. The product of two spaces is incident to each of them. Unity is the highest space, to which all other spaces are incident Nought is the lowest space, being included in all other spaces.* Compare the use of unity here with its use in Chapter V.

The theorems already given will suffice to show that the language and subject matter of the *Ausdehnungslehre* can be utilized in the study of logic. The material for this chapter is taken from Robert Grassmann's *Ausdehnungslehre* (Slettin, 1891). Robert Grassmann aided his brother Hermann in the preparation of the *Ausdehnungslehre* of 1862 and the *Formenlehre* of 1872, and was always deeply interested in the subject.

The article in the Britannica already referred to and Professor Stokes' article in the January (1900) number of The American Mathematical Monthly show that there is considerable diversity of opinion regarding the philosophy underlying the application of mathematics to logic.

INTEGRATION OF ELLIPTIC INTEGRALS.

By G. B. M. ZEER, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

[Concluded from November Number.]

$$\int\frac{d\theta}{\cos^6\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{1}{1-e^2}\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^6\theta}$$

$$-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}=\frac{(8-19e^2+15e^4)}{15(1-e^2)^2}F(e,\ \theta)$$

$$-\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}E(e,\ \theta)+\frac{(8-23e^2+23e^4)}{15(1-e^2)^3}\tan\theta\sqrt{(1-e^2\sin^2\theta)}$$

$$+\frac{\sin\theta}{5(1-e^2)\cos^5\theta}\sqrt{(1-e^2\sin^2\theta)}+\frac{4(1-2e^2)}{15(1-e^2)^2}\cdot\frac{\sin\theta}{\cos^3\theta}\sqrt{(1-e^2\sin^2\theta)}=S_3\ldots\ldots(108).$$

From the foregoing we deduce the following relations :

$$V_2=e^2V_1+U_2,\quad V_3=e^2V_2+U_3.$$

$\therefore$ Generally, $V_n=e^2V_{n-1}+U_n$ ....(109).

$3V_2=2(1+e^2)V_1-e^2V_0-(\cos/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5V_3=4(1+e^2)V_2-3e^2V_1-(\cos\theta/\sin^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)V_n=(2n-2)(1+e^2)V_{n-1}-(2n-3)e^2V_{n-2}$

$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$......................(110).

$$(1-e^2)S_2=R_2-e^2S_1,\quad (1-e^2)S_3=R_3-e^2S_2.$$

$\therefore$ Generally, $(1-e^2)S_n=R_n-e^2S_{n-1}$..........................(111).

$3(1-e^2)S_2=2(1-2e^2)S_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5(1-e^2)S_3=4(1-2e^2)S_2-3e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)(1-e^2)S_n=(2n-2)(1-2e^2)S_{n-1}$

$-(2n-3)e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$........(112).

$3U_2=(2-e^2)U_1-e^2(1-e^2)v_0-(\cos\theta/\sin^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5U_3=(4-e^2)U_2-e^2(1-e^2)V_1-(\cos\theta/\sin^5)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)U_n=(2n-2-e^2)u_{n-1}-e^2(1-e^2)v_{n-2}$

$-(\cos\theta/\sin^{2n-1}\theta)\sqrt{(1-e^2\sin^2)}$ ....(113).

$3(1-e^2)R_2=(2-e^2)R_1-e^2S_0+(\sin\theta/\cos^3\theta)\sqrt{(1-e^2\sin^2\theta)}$,

$5(1-e^2)R_3=(4-3e^2)R_2-e^2S_1+(\sin\theta/\cos^5\theta)\sqrt{(1-e^2\sin^2\theta)}$.

$\therefore$ Generally, $(2n-1)(1-e^2)R_n=[2n(1-e^2)-(2-3e^2)]R_{n-1}$

$-e^2S_{n-2}+(\sin\theta/\cos^{2n-1}\theta)\sqrt{(1-e^2\sin^2\theta)}$ ....(114).

The following integrals come under the above :

$$\int \sqrt{(a^4+2b^2x^2+x^4)}dx, \quad \int \sqrt{(a^4-2b^2x^2+x^4)}dx, \quad \text{where } a<b.$$

Let $b^2+(b^4-a^4)^{\frac{1}{2}}=m^2$, $b^2-(b^4-a^4)^{\frac{1}{2}}=n^2$.

$\therefore$ $(a^4+2b^2x^2+x^4)=(x^2+m^2)(x^2+n^2)$ .... (115).

$(a^4-2b^2x^2+x^4)=(x^2-m^2)(x^2-n^2)$ .... (116).

In (115), let $x=n\tan\theta$. $\therefore$ $dx=n\sec^2\theta d\theta$.

In (116), let $x=m\operatorname{cosec}\theta$. $\therefore$ $dx=-m\operatorname{cosec}\theta\cot\theta d\theta$.

Let $e^2=(m^2-n^2)/m^2$, $k^2=n^2/m^2$.

$$\therefore \int \sqrt{(a^4+2b^2x^2+x^4)}dx=n^2\int\frac{\sqrt{[m^2-(m^2-n^2)\sin^2\theta]}d\theta}{\cos^4\theta}$$

$$=mn^2\int\frac{\sqrt{(1-e^2\sin^2\theta)}d\theta}{\cos^4\theta}=mn^2R_2, \text{ from (102).}$$

$$\int \sqrt{(a^4-2b^2x^2+x^4)}dx=-m^3\int\frac{\cos^2\theta\sqrt{(m^2-n^2\sin^2\theta)}d\theta}{\sin^4\theta}$$

$$=m^3\int\frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^2\theta}-m^3\int\frac{\sqrt{(1-k^2\sin^2\theta)}d\theta}{\sin^4\theta}=m^3(U_1-U_2), \text{ from (97), (101)}$$

Let $(3)=G_1$, $(5)=G_2$, $(10)=G_3$, etc. $(4)=H_1$, $(7)=H_2$, $(13)=H_3$, etc.

$(8)=I_1$, $(15)=I_2$, $(22)=I_3$, etc. $(9)=K_1$, $(16)=K_2$, $(25)=K_3$, etc.

and writing (e, θ) for $(e, \frac{1}{2}\pi)$ we get

$3e^2G_2=2(1+e^2)G_1-G_0$, $5e^2G_3=4(1+e^2)G_2-3G_1$.

$\therefore$ Generally, $(2n-1)e^2G_n=(2n-1)(1+e^2)G_{n-1}-(2n-3)G_{n-2}$ .... (117).

$3e^2H_2=(1-e^2)H_0-2(1-2e^2)H_1$, $5e^2H_3=3(1-e^2)H_1-4(1-2e^2)H_2$.

$\therefore$ Generally, $(2n-1)e^2H_n=(2n-2)(1-e^2)H_{n-2}$

$-(2n-1)(1-2e^2)H_{n-1}$ .... (118).

$I_1=G_1-e^2G_2$, $I_2=G_2-e^2G_3$, $I_3=G_3-e^2G_4$.

$\therefore$ Generally, $I_n = G_n - e^2 G_{n+1}$ ....(119).

$K_1 = (1-e^2)H_1 + e^2 H_2, \quad K_2 = (1-e^2)H_2 + e^2 H_3, \quad K_3 = (1-e^2)H_3 + e^2 H_4.$

$\therefore$ Generally, $K_n = (1-e^2)H_n + e^2 H_{n+1}$ ....(120).

$$\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}} + e^2\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$\therefore \int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} + \frac{1}{e^2}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{1}{e^2}F(e,\ \theta) - \frac{1}{e^2}E(e,\ \theta) + \frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} = T_1 \quad ..(121).$$

$$\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta}{\sqrt{(1-e^2\sin^2\theta)}} - e^2\int\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{\tan\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{1}{e^2(1-e^2)}E(e,\ \theta) - \frac{1}{e^2}F(e,\ \theta) - \frac{\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_1 \quad ....(122).$$

$$\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = T_1 + W_1 = \frac{1}{1-e^2}E(e,\ \theta) - \frac{e^2\sin\theta\cos\theta}{(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_0 \quad ..(123_0).$$

$$\int\frac{\cos^2\theta d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cos^2\theta\cot\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 2(1-e^2)\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-e^2\int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{e^2}\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2(1-e^2)}{e^2}\int\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\cos^2\theta\cot\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}} = \frac{(2-e^2)}{e^4}E(e,\ \theta)$$

$$-\frac{2(1-e^2)}{e^4}F(e,\ \theta) - \frac{1-e^2}{e^2}\cdot\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} = T_2 \quad ..(123).$$

$$\int\frac{\cos^4\theta d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}} = -\frac{\cot\theta\cos^4\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 4(1-e^2)\int\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-3e^2\int\frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$$\therefore \int \frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{2}{3e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{1}{3e^2}\int \frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{3e^2}\int \frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{4(1-e^2)}{3e^2}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\cot\theta\cos^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{(8-17e^2+9e^4)}{3e^6}F(e,\ \theta) - \frac{(8-13e^2+3e^4)}{3e^6}E(e,\ \theta)$$

$$+\frac{(4-6e^2+3e^4)}{3e^4}.\frac{\sin\theta\cos\theta}{\sqrt{(1-e^2\sin^2\theta)}} - \frac{\sin^3\theta\cos\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}} = T_3 \quad .. (124).$$

$$\int \frac{dx}{(a^4+2b^2x^2+x^4)^{\frac{3}{2}}} = \frac{1}{n^2}\int \frac{\cos^4\theta d\theta}{[m^2-(m^2-n^2)\sin^2\theta]^{\frac{3}{2}}} = \frac{1}{n^2 m^3}\int \frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= \frac{T_2}{n^2 m^3} \ldots . (125). \qquad \int \frac{dx}{(a^4-2b^2x^2+x^4)^{\frac{3}{2}}} = -\frac{1}{m^2}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(m^2-n^2\sin^2\theta)^{\frac{3}{2}}}$$

$$= -\frac{1}{m^5}\int \frac{\sin^4\theta d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{2}{m^5}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} - \frac{1}{m^5}\int \frac{\cos^2\theta d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1)$$

$$-\frac{1}{m^5}\int \frac{d\theta}{\cos^2\theta(1-k^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{m^5}(2W_0 - T_1) + \frac{k^2}{m^5(1-k^2)}\int \frac{d\theta}{(1-k^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{1}{m^5(1-k^2)}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-k^2\sin^2\theta)}} = \frac{1}{m^5}\left(\frac{2-k^2}{1-k^2}W_0 - \frac{1}{1-k^2}S_1 - T_1\right) .. (126).$$

In (126) and in the value of $\int \sqrt{(a^4-2b^2x^2+x^4)}dx$, k is written instead of e.

$$\int \frac{\sin^2\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^2\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 2\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+e^2\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \quad \therefore \int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{1}{e^2}\int \frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}} + \frac{2}{e^2}\int \frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} - \frac{\tan\theta\sin^2\theta}{e^2\sqrt{(1-e^2\sin^2\theta)}}$$

$$= \frac{2-e^2}{e^4(1-e^2)}E(e,\ \theta) - \frac{2}{e^4}F(e,\ \theta) - \frac{\sin\theta\cos\theta}{e^2(1-e^2)\sqrt{(1-e^2\sin^2\theta)}} = W_2 \ \ldots\ldots (127).$$

$$\int \frac{\sin^4\theta d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}} = \frac{\tan\theta\sin^4\theta}{\sqrt{(1-e^2\sin^2\theta)}} - 4\int \frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$+3e^2\int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}. \quad \therefore \int \frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}} = \frac{1}{3e^2}\int \frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}$$

$$-\frac{2}{3e^2}\int\frac{d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{1}{3e^2}\int\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}+\frac{4}{3e^2}\int\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$-\frac{\tan\theta\sin^4\theta}{3e^2\sqrt{(1-e^2\sin^2\theta)}}=\frac{8-3e^2-2e^4}{3e^6(1-e^2)}E(e,\ \theta)-\frac{(8+e^2)}{3e^6}F(e,\ \theta)$$

$$-\frac{(1-e^2)\sin\theta\cos\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}-\frac{\sin\theta\cos^3\theta}{3e^4\sqrt{(1-e^2\sin^2\theta)}}=W_3\ \ldots.\ (128).$$

Comparing (3) and (121) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (123) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (12) and (124) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^4\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^{2n-2}\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=\int_0^{\frac{1}{2}\pi}\frac{\cos^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

Comparing (4) and (122) we get,

$$\int_0^{\frac{1}{2}\pi}\frac{\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (6) and (127) we get,

$$3\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

Comparing (11) and (128) we get,

$$5\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$$

$\therefore$ Generally, $(2n-1)\displaystyle\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n-2}\theta\cos^2\theta d\theta}{\sqrt{(1-e^2\sin^2\theta)}}=(1-e^2)\int_0^{\frac{1}{2}\pi}\frac{\sin^{2n}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}.$

$$\int\frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\int\frac{d\theta}{\sin^2\theta\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=V_1+e^2W_0=L_1.$$

$$\int\frac{d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{1-e^2}\int\frac{d\theta}{\cos^2\theta\sqrt{(1-e^2\sin^2\theta)}}-\frac{e^2}{1-e^2}\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=[1/(1-e^2)](S_1-e^2W_0)=O_2.$$

$$\int\frac{d\theta}{\sin^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\int\frac{d\theta}{\sin^4\theta\sqrt{(1-e^2\sin^2\theta)}}+e^2\int\frac{d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}$$

$$=V_2+e^2L_1=L_2.$$

$$\int\frac{d\theta}{\cos^4\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\left(\int\frac{d\theta}{\cos^4\theta\sqrt{(1-e^2\sin^2\theta)}}-e^2O_1\right)\left(\frac{1}{1-e^2}\right)$$

$$=[1/(1-e^2)](S_2-e^2O_1)=O_2.$$

$\therefore$ Generally, $\displaystyle\int\frac{d\theta}{\sin^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=V_n+e^2L_{n-1}=L_n.$

$$\int\frac{d\theta}{\cos^{2n}\theta(1-e^2\sin^2\theta)^{\frac{3}{2}}}=\frac{1}{1-e^2}(S_n-e^2O_{n-1})=O_n.$$

The following general integrals may be of interest. The last term in the right hand member being the one sought.

$$\int\left(\frac{1}{\cos^2\theta}-\frac{1}{\sin^2\theta}\right)\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=\frac{\sec\theta\operatorname{cosec}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$+(2n-1)e^2\int\frac{d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{\cos^{2m-2}\theta d\theta}{\sin^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=-\frac{\cot\theta\cos^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)(1-e^2)\int\frac{\cos^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}-(2m-2n-1)e^2\int\frac{\cos^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{\sin^{2m-2}\theta d\theta}{\cos^2\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}=\frac{\tan\theta\sin^{2m-2}\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}$$

$$-(2m-2)\int\frac{\sin^{2m-2}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}+(2m-2n-1)e^2\int\frac{\sin^{2m}\theta d\theta}{(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\int\frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}+e^2\int\frac{d\theta}{\sin^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$=\int\frac{d\theta}{\sin^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

$$\frac{1}{1-e^2}\int\frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n-1)}}-\frac{e^2}{1-e^2}\int\frac{d\theta}{\cos^{2m-2}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}$$

$$=\int\frac{d\theta}{\cos^{2m}\theta(1-e^2\sin^2\theta)^{\frac{1}{2}(2n+1)}}.$$

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

Apropos of the ending of a century, the following is said to have been written on New Year's Day, 1801, by Theodore Dwight (1764—1846), the brother of Timothy Dwight, the distinguished Congregational divine.

"Precisely 12 o'clock last night
The 18th century took its flight.
Full many a calculating head
Has racked its brains, its ink has shed
To prove by metaphysics fine
A hundred means but ninety-nine;
While at their wisdom others wondered,
But took one more to make a hundred.
Strange at the 18th century's close
While light in beams effulgent glows,
When bright illumination's ray
Has chased the darkness far away,
Heads filled with mathematics lore
Dispute if two and two make four.
Go on, ye scientific sages,
Collect your light a few more ages,
Perhaps as swells the vast amount
A century hence you'll learn to count."

December 10, 1900. M. A. GRUBER.

133. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tennessee.

In Wentworth's Arithmetic he gives a formula $\frac{2}{4}\frac{1}{0}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

No solution of this problem has been received.

135. Proposed by NELSON L. RORAY, Bridgeton, N. J.

If 6 is one-half of 10, what part of 20 is 12? Also what part of 30 is 10?

Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; MARTIN SPINKS, Wilmington, O.; P. S. BERG, B. S., Larimore, N. D.; G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; DANIEL B. NORTHROP, Mandana, N. Y.

The general statement of this problem is, If a is nb, c is dx. What is the value of x? $a:nb=c:dx$.

$\therefore\ x=\dfrac{nbc}{ad}$.

Substituting the numerical values, $x=\dfrac{\frac{1}{2}\times10\times12}{6\times20}=\frac{1}{2}$.

$\therefore$ 12 is one-half of 20, if, etc.

Also $x=\dfrac{\frac{1}{2}\times10\times10}{6\times30}=\frac{5}{18}$.

$\therefore$ 10 is $\frac{5}{18}$ of 30, if, etc.

PROOF. $\frac{1}{2}$ of $10=5$; $\frac{1}{2}$ of $20=10$; and $\frac{5}{18}$ of $30=8\frac{1}{3}$.

$6:5=12:10$; also, $6:5=10:8\frac{1}{3}$.

GEOMETRY.

SOME PROPOSITIONS ON THE REGULAR DODECAHEDRON.

By ALFRED HUME, C. E., D. Sc.. Professor of Mathematics, University of Mississippi, University, Miss.

(1). The edges are parallel, in pairs.

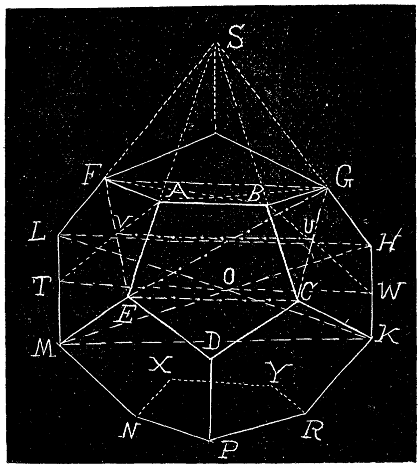
To prove HK parallel to LM.

Since the faces are equal regular pentagons FG and EC are equal.

They are also parallel, each being parallel to AB.

Hence GC and FE are equal and parallel.

Therefore HK, which is parallel to GC, and LM, which is parallel to FE, are parallel.



(2). The planes of the faces are parallel, in pairs.

To prove the plane of face AG parallel to the plane of face NR.

As in (1) GC was proved parallel to FE.

So may FB be proved parallel to MD, and, therefore, parallel to NP.

Similarly, AG and PR are parallel.

Therefore, the planes of faces NR and AG are parallel.

(3). Edges in different faces extending from vertices of the same face are inclined to each other at angles of 60°.

To prove HG and LF inclined to each other at an angle of 60°.

EA and CB produced intersect at S.

From the similar triangles SAB and SEC, $SA:AB=SE:EC$.

Representing an edge of the dodecahedron by a, and a diagonal of a face by d, this proportion gives $SA=\dfrac{a^2}{d-a}$.

But $d=\dfrac{a}{2}[\sqrt{5}+1]$. Therefore, $SA=\dfrac{a}{2}[\sqrt{5}+1]=d$.

That is, CB produced meets EA produced at a point distant d from A.

Similarly LF produced meets EA produced at a point distant d from A; hence, at S.

In this way it can be shown that HG produced passes through S.

Hence LF and HG are in the same plane; and triangle SFG is equilateral, each side being equal to d.

Therefore LF and HG are inclined to each other at an angle of 60°.

(4). If the ten diagonals of two non-adjacent, non-parallel, faces being grouped in pairs, both faces being represented in every pair, (a) the diagonals of one pair are parallel, (b) those of each of two pairs meet, when produced,

at angles of $60°$, and (c) those of each of two other pairs are inclined to each other at angles of $36°$.

Consider the faces BK and AM.

(a). CG is parallel to EF (proved in the demonstration of (1)).

(b). KB and MA lie in the same plane.

For AB is parallel to FG; and FG is parallel to LH, since FG divides the sides of the triangle SLH proportionally; and LH is parallel to MK, since LM and HK and equal and parallel.

Hence AB and MK are parallel and determine a plane.

KB and MA are parallel to HG and LF, respectively.

Therefore, when, produced, they meet at an angle of $60°$.

Similarly with HB and LA.

(c). KG and MF are in the same plane, since FG and MK are parallel.

And they are parallel to CB and EA, respectively.

The angle between them, therefore, equals angle ASB, or $36°$.

So with HC and LE.

(5). To express the diagonals in terms of a and d.

One of the shortest is EG, one of the longest is LK, and a medium diagonal is LH.

Denote these by s, l, and m, respectively.

LH is a side of the equilateral triangle SLH.

Hence, $m=SF+FL=\frac{a}{2}[\sqrt{5}+1]+a$

$$=\frac{a}{2}[\sqrt{5}+3]=\frac{1}{a}\left[\frac{a}{2}[\sqrt{5}+1]\right]^2=\frac{d^2}{a}.$$

To determine s and l it is necessary to show that $LMKH$ is a rectangle, and $FECG$ a square.

The plane bisecting LM at right angles contains all points equidistant from L and M, and is perpendicular to all lines parallel to LM.

Therefore this plane will pass through the middle point of FE, through points A and B, and, hence, the line AB, through the middle points of GC and HK, intersecting the planes LK and FC in the lines TW and UV, respectively.

Since AB is parallel to FG and LH, it is parallel to planes FC and LK, and, hence, to UV and TW.

FG, UV, LH, and TW, then, are parallel, and since FE and LM are respectively perpendicular to UV and TW, they are perpendicular to FG and LH respectively.

It follows that FC is a square and LK a rectangle.

From right triangle ECG, $s=\frac{a\sqrt{2}}{2}[\sqrt{5}+1]=\sqrt{2}\,d.$

From right triangle LMR, $l=\frac{a\sqrt{3}}{2}[\sqrt{5}+1]=\sqrt{3}\,d.$

This result may be written $\sqrt{\frac{a}{2}[3+\sqrt{5}]3a}=\sqrt{3am}.$

It may be remarked that d is a mean proportional between a and m, l is a mean proportional between $3a$ and m, and l is the diagonal of a cube whose face-diagonal is s and edge d.

(6). To determine a point equidistant from all the vertices.

LK bisects MH at O.

In the rectangle of which GH and MN are opposite sides, GN bisects MH, therefore passes through O and is itself bisected at O.

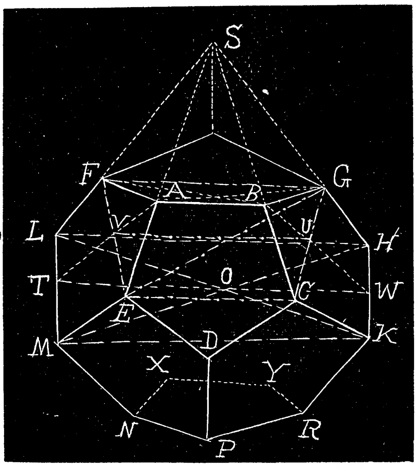
So with all the longest diagonals.

Hence O is equidistant from all the vertices.

COROLLARY 1. O is equidistant from all the edges.

COROLLARY 2. O is equidistant from all the faces.

(7). To compute the volume.



This could be done by conceiving the dodecahedron composed of twelve equal pyramids with O as their common vertex and the faces for bases. But another method will be adopted.

The dodecahedron is composed of a cube one of whose faces is $EFGC$ and six equal truncated triangular prisms, one of which has AB, FG, and EC for its lateral edges. A right section of this truncated prism is a triangle with base equal to EF, and altitude, the perpendicular from A to UV. Since $ABXY$ is a rectangle equal to $LMKH$, this perpendicular is one-half the difference between $AX(=LH)$ and an edge of the cube, and, therefore, equals $\frac{1}{2}a$.

Hence the area of the right section is $\frac{a^2}{8}[\sqrt{5}+1]$, and the volume of the truncated prism is $\frac{a^2}{8}[\sqrt{5}+1]\frac{a+a[\sqrt{5}+1]}{3}$.

The sum of the volumes of six such solids is $\frac{a^3}{4}[7+3\sqrt{5}]$.

Volume of cube$=\frac{a^3}{4}[8+4\sqrt{5}]$.

Volume of dodecahedron$=\frac{a^3}{4}[15+7\sqrt{5}]$.

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134. Proposed by J. C. GREGG, A. M., Superintendent of Schools, Brazil, Ind.

If $ABCD$ is a quadrilateral circumscribing a circle, show that the line joining the middle points of the diagonals AB, CD passes through the center of the circle.

Solution by HARRY S. VANDIVER, Bala, Penna.

We will use quadrilinear coördinates denoting the equation of the four

sides $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, and the *lengths* of the corresponding sides by a, b, c, and d. Let radius of circle be r. Then the equation of the line joining the middle points of the diagonals is $a\alpha-b\beta+c\gamma-d\delta=0$....(1) (Cf. Salmon's *Conics*, page 54, Ex. 5).

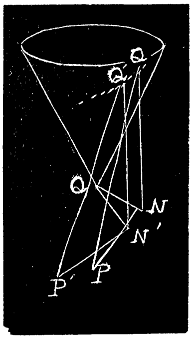
Putting $\alpha=\beta=\gamma=\delta=r$ we obtain $r(a-b+c-d)=0$, which is satisfied since $a+c=b+d$.

137. Proposed by **J. W. YOUNG**, Oliver Graduate Scholar in Mathematics, Cornell University, Ithaca, N. Y.

A right cone has its vertex in a horizontal plane, its axis being perpendicular to the plane. A string has one extremity attached to a point on the cone. The other extremity, P, of the string is kept in the plane, and the string is then wound around the cone, without being allowed to slip. Show that the spiral generated by P cuts all straight lines through the vertex at the same angle.

Solution by the PROPOSER.

Let P, P' be two points on the spiral; Q, Q' the corresponding points in the path of the string around the cone; N, N' the points where the perpendiculars from Q, Q' to the plane through the vertex O of the cone, cut the plane.



The right-angled triangles QNO. $Q'N'O$ have the angles QON and $Q'ON'$ equal; hence they are similar.

$$\therefore \frac{QN}{ON}=\frac{Q'N'}{ON'}\ldots.(1).$$

Again, since the string must not slip, it makes a constant angle with the plane.

$\therefore \triangle QNP$ is similar to $\triangle Q'N'P'$.

$$\therefore \frac{PN}{QN}=\frac{P'N'}{Q'N'}\ldots.(2).$$

From (1) and (2),

$$\therefore \frac{PN}{ON}=\frac{P'N'}{ON'}\ldots.(3).$$

But the triangles ONP and $ON'P'$ are right-angled at N and N' (PN, $P'N'$ being the projections to tangents to the circular cone). From (3),

$\therefore \triangle ONP$ is similar to $\triangle ON'P'$.

$\therefore \measuredangle OPN=\measuredangle OP'N'$.

Observing that PN and $P'N'$ are normals to the spiral, the last equation states that the normals make a constant angle with rays through O. Q. E. D.

AVERAGE AND PROBABILITY.

90. Proposed by **WALTER H. DRANE**, Graduate Student, Harvard University, Cambridge, Mass.

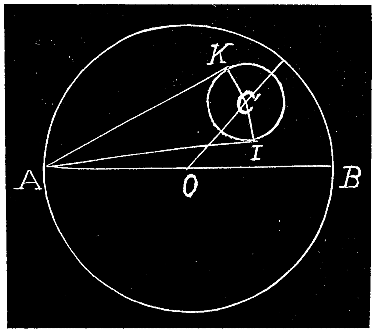
During a heavy rain storm a circular pond is formed in a circular field. If a man undertakes to cross the field in the dark, what is the chance that he will walk into the pond? [From *Byerly's Integral Calculus.*]

I. Solution by **B. F. FINKEL, A. M., M. Sc.**, Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Let O be the center of the circular field, and R its radius; C the center of

the circular pond and z its radius; $x=OC$, the distance from the center of the field to the center of the pond; $\angle BOC=\theta$, $\angle IAC=\phi$.

Then $AC=(R^2+x^2+2Rx\cos\theta)^{\frac{1}{2}}$ and $\phi=\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)$.



Then the probability that the man will walk into the pond for any particular value of ϕ is

$$\frac{2\phi.AK}{\pi AK}=\frac{2\phi}{\pi}.$$

The limits of z are 0 and $R-x$; of x, 0 and R; and of θ, 0 and π, and doubled.

Hence, the probability that the man walks into the pond is

$$p=\frac{2}{\pi}\frac{\int_0^\pi\int_0^R\int_0^{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)d\theta.2\pi x dx.dz}{\int_0^\pi\int_0^R\int_0^{R-x}d\theta.2\pi x dx.dz}$$

$$=\frac{12}{\pi^2R^3}\int_0^\pi\int_0^R\int_0^{R-x}\sin^{-1}\left(\frac{z}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)d\theta.xdx.dz$$

$$=\frac{12}{\pi^2R^3}\int_0^\pi\int_0^R\left[z\sin^{-1}\left(\frac{z}{(R^2+x^2+2Rx\cos\theta)^{\frac{1}{2}}}\right)\right.$$

$$\left.+\sqrt{1-\frac{z^2}{(R^2+x^2+2Rx\cos\theta)}}\right]_0^{R-x}d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^\pi\int_0^R\left[(R-x)\sin^{-1}\left(\frac{R-x}{\sqrt{(R^2+x^2+2Rx\cos\theta)}}\right)+2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta\right.$$

$$\left.-\sqrt{(R^2+x^2+2Rx\cos\theta}\right]d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^\pi\int_0^R(R-x)\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)+\frac{12}{\pi^2R^3}\int_0^\pi\int_0^R[2\sqrt{(Rx)}\cos\tfrac{1}{2}\theta$$

$$-\sqrt{(R^2+x^2+2Rx\cos\theta)}]d\theta.xdx$$

$$=\frac{12}{\pi^2R^3}\int_0^\pi\left(\left[\tfrac{1}{6}x^2\tan^{-1}\left(\frac{R-x}{2\sqrt{(Rx)}\cos\frac{1}{2}\theta}\right)\right]_0^R\right.$$

$$\left.+\tfrac{1}{6}\int_0^R\frac{x^{\frac{3}{2}}R^{\frac{1}{2}}(R+x)(3R-2x)\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}dx\right)d\theta$$

$$+\frac{12}{\pi^2R^3}\int_0^\pi\Big[\tfrac{4}{5}R^{\frac{1}{2}}x^{\frac{5}{2}}\cos\tfrac{1}{2}\theta-\tfrac{1}{3}(R^2+x^2+2Rx\cos\theta)^{\frac{3}{2}}+\tfrac{1}{2}R\cos\theta(x+2R\cos\theta)$$

$$\sqrt{(R^2+x^2+2Rx\cos\theta)}+\tfrac{1}{2}R^3\sin^2\theta\cos\theta\log[\sqrt{(R^2+x^2+2Rx\cos\theta)}+x+R\cos\theta]\Big]_0^R d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\int_0^\pi\frac{x^{\frac{3}{2}}R^{\frac{1}{2}}(R+x)(3R-2x)\cos\frac{1}{2}\theta}{R^2+x^2+2Rx\cos\theta}dx\,d\theta+\frac{12}{\pi^2R^3}\int_0^\pi\frac{R^3}{30}\Big[24\cos\tfrac{1}{2}\theta$$

$$-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta+15\sin^2\theta\cos\theta\log\left(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)\Big]d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\left[\frac{1}{\sqrt{-1}}x(3R-2x)\tan^{-1}\left(\frac{2\sqrt{(-Rx)}}{R+x}\sin\tfrac{1}{2}\theta\right)\right]_0^\pi dx+\frac{2}{5\pi^2}\int_0^\pi\Big[24\cos\tfrac{1}{2}\theta$$

$$-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta+15\sin^2\theta\cos\theta\log\left(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)\Big]d\theta$$

$$=\frac{2}{\pi^2R^3}\int_0^R\frac{1}{\sqrt{-1}}\left[x(3R-2x)\tan^{-1}\left(\frac{2\sqrt{(-Rx)}}{R+x}\right)\right]dx-\frac{32}{\pi^2}$$

$$=\frac{2}{\pi^2R^3}\left(\left[\frac{x^2}{6\sqrt{-1}}(9R-4x)\tan^{-1}\left(\frac{2\sqrt{(-Rx)}}{R+x}\right)\right]_0^R\right.$$

$$\left.-\frac{\sqrt{(-R)}}{6\sqrt{(-1)}}\int_0^R\frac{x^{\frac{3}{2}}(9R-4x)}{R-x}dx\right)-\frac{32}{\pi^2}=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}(\sqrt{-1})$$

$$-\frac{1}{3\pi^2R^{\frac{5}{2}}}\int_0^R(4x^{\frac{3}{2}}-5Rx^{\frac{1}{2}}-5R^2x^{-\frac{1}{2}}+\frac{5R^2}{\sqrt{x}\,(R-x)})dx-\frac{32}{\pi^2}$$

$$=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}(\sqrt{-1})-\frac{1}{3\pi^2R^{\frac{5}{2}}}\Big[\tfrac{8}{5}x^{\frac{5}{2}}-\tfrac{10}{3}Rx^{\frac{3}{2}}-10R^2x^{\frac{1}{2}}$$

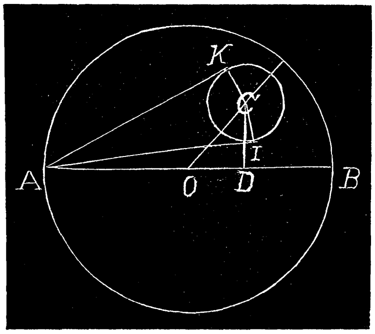
$$+\frac{5R^{\frac{3}{2}}}{\sqrt{-1}}\tan^{-1}\sqrt{\left(-\frac{x}{R}\right)}\Big]_0^R-\frac{32}{\pi^2}=\frac{176}{45\pi^2}-\frac{32}{15\pi^2}=\frac{16}{9\pi^2}.$$

J. M. Colaw, M. E. Graber, and Walter H. Drane get the result $p=l/L$, where l is the circumference of the pond, and L the circumference of the field. But the circumference of the pond varies from 0 to $2\pi R$ where R is the radius of the field. Assuming as they do, that the chance of the man walking into the pond is the totality of all lines crossing both the field and pond divided by the totality of all lines crossing the field, the result l/L is not satisfactory, since l is variable, as already stated.

The problem, of course, is indefinite, since the law of variation of the several events are not definitely stated, and thus as many different results may be obtained as there are possible interpretations of the meaning of the problem.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let O be the center of the field, radius r; C the center of the pond, radius c. The chance p, of walking into the pond is $\angle KAI/\pi=\varphi/\pi$.



Let $(x-a)^2+(y-b)^2=c^2$ be the equation to the pond; $y=m(x+r)$, the equation to a line meeting the pond.

When $y=m(x+r)$ is tangent to the pond we have

$$\tan BAE=m_2=-\frac{b(a+r)+c\sqrt{b^2-c^2+(a+r)^2}}{c^2-(a+r)^2}$$

$$\tan BAF=m_1=-\frac{b(a+r)-c\sqrt{b^2-c^2+(a+r)^2}}{c^2-(a+r)^2}$$

$$\therefore\ \tan EAF=\frac{m_1-m_2}{1+m_1m_2}=\frac{2c\sqrt{[b^2-c^2+(a+r)^2]}}{[b^2-c^2+(a+r)^2]-c^2}=\tan\varphi.$$

$$\therefore\ \varphi=\tan^{-1}\left(\frac{2c\sqrt{[b^2-c^2+(a+r)^2]}}{[b^2-c^2+(a+r)^2]-c^2}\right)=2\tan^{-1}\left(\frac{c}{\sqrt{[b^2-c^2+(a+r)^2]}}\right).$$

If $a=\rho\cos\theta$, $b=\rho\sin\theta$, $\varphi=2\tan^{-1}\left(\dfrac{c}{\sqrt{(\rho^2+r^2+2\rho r\cos\theta)}}\right)$.

$$\therefore\ p=\frac{\frac{1}{\pi}\int_0^\pi\int_0^r\int_0^{r-\rho}\varphi\rho\, d\theta\, d\rho\, dc}{\int_0^\pi\int_0^r\int_0^{r-\rho}\rho\, d\theta\, d\rho\, dc}=\frac{6}{\pi^2r^3}\int_0^\pi\int_0^r\int_0^{r-\rho}\varphi\rho\, d\theta\, d\rho\, dc$$

$$=\frac{12}{\pi^2r^3}\int_0^\pi\int_0^r\left[(r-\rho)\tan^{-1}\left(\frac{r-\rho}{2\sqrt{(r\rho)}\cos\frac{1}{2}\theta}\right)+2\sqrt{(r\rho)}\cos\tfrac{1}{2}\theta\right.$$
$$\left.-\sqrt{(\rho^2+r^2+2r\rho\cos\theta)}\right]\rho\, d\theta\, d\rho.$$

$$\frac{12}{\pi^2r^3}\int_0^\pi\int_0^r[2\sqrt{(r\rho)}\cos\tfrac{1}{2}\theta-\sqrt{(\rho^2+r^2+2r\rho\cos\theta)}]\rho\, d\theta\, d\rho$$

$$=-\frac{2}{5\pi^2}\int_0^\pi\left[24\cos\tfrac{1}{2}\theta-140\cos^3\tfrac{1}{2}\theta+120\cos^5\tfrac{1}{2}\theta+10-15\cos^2\theta\right.$$
$$\left.+15\sin^2\theta\cos\theta\log\left(\frac{1+\cos\frac{1}{2}\theta}{\cos\frac{1}{2}\theta}\right)\right]d\theta=-\frac{32}{15\pi^2}.$$

$$\frac{12}{\pi^2r^3}\int_0^\pi\int_0^r(r-\rho)\tan^{-1}\left(\frac{r-\rho}{2\sqrt{(r\rho)}\cos\frac{1}{2}\theta}\right)\rho\, d\theta\, d\rho$$

$$=\frac{2}{\pi^2r^3}\int_0^\pi\int_0^r\frac{\rho\sqrt{(r\rho)}(r+\rho)(3r-2\rho)\cos\frac{1}{2}\theta d\rho d\theta}{\rho^2+r^2+2r\rho\cos\theta}$$

$$=\frac{2}{\pi^2r^3\sqrt{-1}}\int_0^r\rho(3r-2\rho)\tan^{-1}\left(\frac{2\sqrt{(-r\rho)}}{r+\rho}\right)d\rho.$$

$$\frac{2}{\pi^2r^3\sqrt{-1}}\int_0^r\rho(3r-2\rho)\tan^{-1}\left(\frac{2\sqrt{(-r\rho)}}{r+\rho}\right)d\rho=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{-1}$$

$$-\frac{1}{3\pi^2r^3}\int_0^r\frac{\sqrt{(r\rho)}(9r\rho-4\rho^2)d\rho}{r-\rho}=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{-1}$$

$$-\frac{1}{3\pi^2r^3}\left[\tfrac{8}{5}\rho^2\sqrt{(r\rho)}-\tfrac{10}{3}r\rho\sqrt{(r\rho)}-10r^2\sqrt{(r\rho)}+\frac{5r^3}{\sqrt{-1}}\tan^{-1}\sqrt{-\frac{\rho}{r}}\right]_0^r$$

$$=\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{(-1)}+\frac{176}{45\pi^2}-\frac{5}{3\pi^2\sqrt{-1}}\tan^{-1}\sqrt{(-1)}=\frac{176}{45\pi^2}.$$

$$\therefore\ p=\frac{176}{45\pi^2}-\frac{32}{15\pi^2}=\frac{16}{9\pi^2}.$$

EDITORIALS.

The Summer Meeting, 1901, of the American Mathematical Society will be held at Cornell University.

Several Complete sets of the MONTHLY are wanted. Any reader possessing a complete set in good condition and desiring to sell it, should write to the editor stating the price desired.

The MONTHLY is mailed on the 28th of each month, and should reach the greater part of its readers within five days after it is mailed. Subscribers should notify us promptly of any change of address, or of any failure to receive their copies.

Dr. G. A. Miller has just been awarded the prize of $260. from the Cracow Academy of Austria for his work in the Theory of Groups. We rejoice with Dr. Miller in this well merited recognition of his work. Dr. Miller is a young mathematician, yet his contributions to *The American Journal of Mathematics*, *The Annals of Mathematics*, *The Messenger of Mathematics*, *The Proceedings of the London Mathematical Society*, *The Transactions of the American Mathematical Society*, the MONTHLY, and many other mathematical journals in both hemispheres, are very numerous.

Preliminary Announcement of the Courses of Study for the Summer Session, 1901, of Cornell University, has just been issued. During the session Dr. G. A. Miller will offer courses in Advanced Integral Calculus, History of Mathematics, and Introduction to the Theory of Groups and the Theory of Numbers.

BOOKS AND PERIODICALS.

Elements of Algebra. By James M. Taylor, A. M., LL. D., Professor of Mathematics, Colgate University. 8vo. Half Leather. 461 pages. Boston: Allyn & Bacon.

This book on Algebra is in keeping with the very excellent work on the Elements of the Calculus by the same author. Professor Taylor aims in this book at simplicity in method of presentation and at a natural and logical sequence in the series of steps which lead the student from his arithmetical experiences through his algebra, and in this, I think, he has succeeded admirably. Special attention is given to factoring, as it is the fundamental principal in the solution of quadratic and higher equations. The student who receives his preparatory mathematical training in a book of this character will have nothing to unlearn as he advances in his mathematical course. B. F. F.

The Elements of Astronomy. By Sir Robert Ball, LL. D., F. R. S., Lowndean Professor of Astronomy and Geometry in the University of Cambridge, and formerly Astronomer Royal of Ireland. 8vo. Cloth, viii+183 pages. Price, 80 cents. London and New York: The Macmillan Co.

The author, briefly but interestingly, traces the early beginnings of Astronomy in the introduction. He then devotes Chapter I. to Diurnal Motion; Chapter II.. The Sun; Chapter III., The Apparent Motion of the Sun; Chapter IV., The Moon; Chapter V., Gravitation; Chapter VI., Mercury and Venus; Chapter VII., Mars: Chapter VIII., The Asteroids; Chapter IX., Jupiter; Chapter X., Saturn; Chapter XI., Uranus and Neptune; Chapter XII., Comets; Chapter XIII., Shooting Stars; Chapter XIV., Stars and Nebulæ; Chapter XV., The Causes Affecting the Apparent Places of the Stars. There are eleven excellent plates in addition to numerous diagrams. For a brief, scientific exposition of Astronomy, this little book is superior. B. F. F.

The Principles of Mechanics. An Elementary Exposition for the Students of Physics. By Frederick Slate, Professor of Physics in the University of California. Part I. 12mo. Cloth, x+299 pages. Price, $1.90. New York: The Macmillan Co.

"The material contained in these chapters has taken on its present form gradually, by a process of recasting, and sifting. The ideas guiding that process have been three: First, to select the subject matter with close reference to the needs of college students; second, to bring the instruction into adjustment with the actual stage of their training; and, third, to aim continually at treating Mechanics as a system of organized thought, having a clearly recognizable culture value."—*Preface.* It is great satisfaction to note the emphasis that modern writers on various branches of mathematics are laying on the culture value of the study of mathematics. The work before us is one worthy the consideration of teachers desiring a first-class text on Mechanics. B F. F.

Calculus with Applications. An Introduction to the Mathematical Treatment of Science. By Ellen Hayes, Professor of Applied Mathematics in Wellesley College. 8vo. Cloth, vii+162 pages. Price, $1.20. Boston and Chicago: Allyn & Bacon.

This little treatise presents in an admirable way the practical application of the calculus to some of the problems of the natural sciences. To those who wish, for the purpose of culture, to know in a simple and direct way, what the Calculus is and what it is for and also for those who are engaged in work in Chemistry, Astronomy, Economics, etc., this book is well adapted. B. F. F.

Differential and Integral Calculus. With Applications. For Colleges, Universities, and Technical Schools. By E. W. Nichols, Professor of Mathematics in the Virginia Military Institute, and author of Nichols's Analytical Geometry. 8vo. Half Leather. xi+394 pages. Boston and Chicago: Allyn & Bacon.

This book is based upon the methods of "limits" and of "rates." Both methods are valuable; the former, however, has the advantage of greater rigor. Among the very commendable features of the book are: First, a large amount of explanations; second, clear and simple demonstrations of principles; third, geometrical, mechanical, and electrical applications; fourth, historical notes at the heads of chapters, giving a brief account of the discovery and development of the subject of which it treats; and fifth, foot notes calling attention to topics of special historic interest. B. F. F.

Esercizi ed Applicazioni di Trigonometria Piana, con 400 Escercizi e Problemi Propositi. Dal Prof. Christoforo Alasia. Milano, Italia: Ulrico Hoepli.

This book is a 12mo. cloth binding, containing 291 pages. It treats of the elements of Plane Trigonometry. Also some space is devoted to the discussion of "limits," "maxima" and "minima," and simultaneous trigonometric equations. There are thirty diagrams and 400 exercises. B. F. F.

Plane Trigonometry. By W. P. Durfee, Professor of Mathematics in Hobart College. 105 pages. Price, 80 cents. Boston: Ginn & Co. 1900.

This work covers the essential parts of Plane Trigonometry. The preliminary chapter on Computation is valuable. Stress is laid on the value of graphic methods in obtaining rough estimates; the limits of accuracy are pointed out, and the necessity of frequent checking is insisted on. The demonstrations are brief and they are illustrated by numerous exercises. The book admirably meets the needs of schools desiring a short course which is at the same time accurate. J. M. C.

Complete Trigonometry. By Webster Wells. S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 171 pages. Price, 90 cents. Boston: D. C. Heath & Co. 1900.

The present volume is a revision of the author's Essentials of Trigonometry. Many improvements have been made, notably in the proofs of several of the functions, in the general demonstration of the formulae, in the solution of the right triangles by natural functions, and in the addition of a large number of examples. This is a brief but most satisfactory text. J. M. C.

Famous Geometrical Theorems and Problems. By William W. Rupert, C. E., Superintendent of Schools, Pottstown, Pa. 27 pages. Price, 10 cents. Boston: D. C. Heath & Co. 1900

This is No. 1 in the series of Heath's Mathematical Monographs, issued under the general editorship of Professor Webster Wells. It contains the history of several famous

theorems with demonstrations under each case. Twenty-six independent demonstrations are given of the Pythagorean theorem. The book will give much pleasure to teachers interested in the subject. J. M. C.

Mensuration. By S. A. Furst. 71 pages. Price, 50 cents. Harrisburg, Pa.: R. L. Myers & Co. 1899.

This little volume is intended to be supplemental to text-books both on arithmetic and geometry. The special feature of the work is the application of the prismoidal formula to finding the volume of solids. The work has many interesting problems and will doubtless meet with approval. J. M. C.

The Hall Arithmetics, (1) Elementary ; (2) Complete. By Frank H. Hall. 248 and 448 pages. Prices, 35 cents and 60 cents. New York and Chicago: The Werner School Book Co. 1899.

The prominent features of the Werner Arithmetics have been preserved in this series, but better adaption for use in ungraded schools has been secured. The books are progressive and practical in character, and abundant in the supply of concrete problems. In the treatment prominence is given the "magnitude idea," and the elements of algebra and geometry have been judiciously introduced. J. M. C.

New Practical and *New Higher Arithmetics.* By A. W. Rich, Ph., B., Associate Professor of Mathematics in the Iowa State Normal School. 222 and 320 pages. Prices, 50 cents and 75 cents, respectively. Chicago: A. Flanagan Co. 1900.

The *Practical* is intended for use in grammar grades. Special features are a set of tables and drills for mental work; a presentation of definitions and principles in compact form; model problem solutions; and a great variety of test work. In the *Higher* book the mathematical signs are systematically presented, the model solution feature is carried forward, and larger place is given to definitions and principles. Numerous exercises and problems are given in all the different subjects. J. M. C.

Hornbrook's Grammar School Arithmetic. By A. R. Hornbrook, A. M. 416 pages. Price, 65 cents. American Book Co. 1900.

This book is designed for the last four years of grammar school work, and aims to develop in the pupil a ready skill in dealing with numbers. In many respects this book differs from the ordinary texts. Much use is made of constructive work with simple geometric forms. The book will repay examination. J. M. C.

We are indebted to Prof. Alexander Macfarlane, D. Sc., LL. D., Lecturer on Mathematical Physics at Lehigh University, for a copy of *Space-Analysis*, a brief of twelve lectures on the George Leib Harrison Foundation, delivered in College Hall, University of Pennsylvania, February 5 to March 2, 1900.

ERRATA.

[Due to errors in the copy.]

Vol. VII, No. 11 (November, 1900), page 240, in the diagram,

for $\frac{\partial W}{\partial y}dy$, read $\frac{\partial W}{\partial y}dy=\delta$; for $i\frac{\partial V}{\partial y}dy$, read $-i\frac{\partial V}{\partial y}dy$; for idy, read $-idy$.

Page 241, line 7 from top,

$$\text{for } \frac{\partial V}{\partial y}dy=\frac{\partial U}{\partial x}dx, \text{ read } \frac{\partial V}{\partial y}=\frac{\partial U}{\partial x}.$$